

Bödeker's Effective Theory: From Langevin Dynamics to Dyson–Schwinger Equations

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The dynamics of weakly coupled, non-abelian gauge fields at high temperature is non-perturbative if the characteristic momentum scale is of order $|\mathbf{k}| \sim g^2 T$. Such a situation is typical for the processes of electroweak baryon number violation in the early Universe. Bödeker has derived an effective theory that describes the dynamics of the soft field modes by means of a LANGEVIN equation. This effective theory has been used for lattice calculations so far [18, 19]. In this work we provide a complementary, more analytic approach based on DYSON–SCHWINGER equations. Using methods known from stochastic quantisation, we recast Bödeker's LANGEVIN equation in the form of a field theoretic path integral. We introduce gauge ghosts in order to help control possible gauge artefacts that might appear after truncation, and which leads to a BRST symmetric formulation and to corresponding Ward identities. A second set of Ward identities, reflecting the origin of the theory in a stochastic differential equation, is also obtained. Finally DYSON–SCHWINGER equations are derived.

I. INTRODUCTION

Lattice calculations are an ideal tool to extract with a minimum of theoretical prejudice a specific piece of information from a given theory. However, in a sense, they are kind of a ‘black box’ that give the answer but hide the way *how* that answer comes about.

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The aim of this work is to provide a complementary, more analytic approach to the non-perturbative physics encoded in Bödeker's effective theory [1]. The emphasis thereby lays not primarily on the accuracy of the results where it is hardly possible to beat the lattice calculations. Our aim is to provide a tool for a deeper understanding of what is really going on in the non-perturbative sector of hot non-abelian gauge theory and during creation of baryon number. In particular, it allows for an analytic study of the sphaleron rate [2, 3]

$$\Gamma \equiv \lim_{V \rightarrow \infty} \lim_{t \rightarrow \infty} \frac{\langle (N_{\text{CS}}(t) - N_{\text{CS}}(0))^2 \rangle}{Vt} \quad (1)$$

Such a deeper understanding is not only important for baryogenesis. Magnetic screening and the corresponding identification of a magnetic mass are of quite general theoretical interest with applications also in the field of the quark–gluon plasma [4–7].

We base our analysis on Bödeker's effective theory, despite the fact that Bödeker has also derived a generalised BOLTZMANN–LANGEVIN equation which is valid to all orders in $[\log(1/g)]^{-1}$ [9], and of which Bödeker's effective theory is merely the leading logarithmic approximation. We choose this approximation because the more general BOLTZMANN–LANGEVIN equation not only is far more complicated, but is also not renormalisable by power counting [10]. The effective theory on the other hand is ultraviolet finite, and is known to still be valid at next-to-leading logarithmic order provided one uses the next-to-leading logarithmic order colour conductivity σ [11].

The key idea of this work is rather simple. Bödeker's effective theory has the form of a LANGEVIN equation. It is well-known from stochastic quantisation that a LANGEVIN equation can be recast in the form of a path integral [15–17]. This path integral then can be reinterpreted as the functional integral formulation of an Euclidean quantum field theory with some ‘strange’ action. In this way, one gains access to all the powerful methods developed in QFT. Specifically, it is possible to derive the DYSON–SCHWINGER equations of the theory, offering an approach to the non-perturbative sector that is independent from, and complementary to, the existing lattice studies.

On the way to this goal a couple of obstacles have to be overcome. These are mostly related to the peculiar role played by gauge invariance in the context of stochastic quantisation and Bödeker's effective theory. A thorough understanding of this role proves to be essential in pursuing our aim.

The outline of this work is as follows. Section II is devoted to the transcription of

Bödeker's theory in path integral form. From this path integral one could proceed to derive the DYSON–SCHWINGER equations, and in principle, could gain access to the non-perturbative sector of the theory. At the end of the day however, one will be forced to rely on a certain truncation scheme to extract any concrete results from the equations. This truncation may introduce a possible gauge dependence and thus may render the results worthless. To keep control over the gauge dependence, it is therefore necessary to generalise Bödeker's equation from $A_0 = 0$ gauge to a more general class of gauges before applying the formalism.

Gauge fixing in a stochastic differential equation is quite delicate. One has to make sure not to destroy the MARKOVIAN nature of the equation. Applying methods developed in the context of stochastic quantisation [12], we introduce a gauge fixing term into Bödeker's equation thereby achieving the desired upgrade to a general class of flow gauges.

In Section III we argue that any physically reasonable truncation of the DYSON–SCHWINGER equations requires the introduction of *gauge* ghosts. In the full, untruncated theory gauge ghosts are not necessary, which is generally true in stochastic quantisation [12, 16]. As was shown in Ref. [12], gauge ghosts *can* be introduced in stochastic quantisation in order to establish a gauge BRST symmetric formulation.

We carry out this program in the case of Bödeker's theory and derive the WARD–TAKAHASHI identities corresponding to the gauge BRST symmetry of the action. These should be respected by the truncations to be used.

The gauge Ward identities are not the only restrictions to be observed. A second class of Ward identities exist, that are related to the characteristic structure of the theory reflecting its origin in a stochastic differential equation. This characteristic structure can as well be expressed in the form of a BRST symmetry by introducing another kind of ghost fields, referred to as equation of motion (EOM) ghosts in this work. Introducing the gauge ghosts, however, destroys this second stochastic BRST symmetry. Nevertheless, it does not change the physical contents of the theory. The stochastic BRST symmetry is only an elegant way to express this structure. By directly referring to the underlying physics, it is still possible to derive the corresponding stochastic Ward identities. They provide a second set of restrictions to be imposed on the truncations.

In Section IV we derive the DYSON–SCHWINGER equations of Bödeker's effective theory. In combination with the gauge and stochastic Ward identities of Section III, this constitutes

an independent approach to the non-perturbative dynamics of the soft, non-abelian gauge fields encoded in Bödeker's effective theory.

In Section V we summarise and discuss our results. Appendix A shows the explicit calculation of some of the Jacobians encountered in this work. We have included this Appendix in order to make our presentation more self-contained. The FEYNMAN rules corresponding to our field theoretic transcription of Bödeker's effective theory are listed in Appendix B. Finally, in Appendix C, we present explicit identities for the lower n-point function following from the general Ward identities.

II. PATH INTEGRAL FORMULATION OF BÖDEKER'S THEORY

A. Transcription to a Path Integral in $A_0 = 0$ Gauge

According to Bödeker's effective theory the dynamics of the soft modes of the gauge field is described to leading logarithmic order by the LANGEVIN equation [1]

$$\mathbf{D}^{ab} \times \mathbf{B}^b + \sigma \dot{\mathbf{A}}^a = \zeta^a \quad (2)$$

which is written in $A_0 = 0$ gauge and where ζ is a gaussian white noise stochastic force. The stochastic force field incorporates the influence of higher momentum modes and has the correlator

$$\langle \zeta^{ai}(t, \mathbf{x}) \zeta^{bj}(t', \mathbf{x}') \rangle = 2\sigma T \delta^{ij} \delta^{ab} \delta(t - t') \delta^{D-1}(\mathbf{x} - \mathbf{x}') \quad (3)$$

reflecting its gaussian white noise character. Here and in the following, the number of spacial dimensions is $D - 1 = 3$, however, we leave D unspecified to allow for dimensional regularisation later. The only physical parameters entering Eqs. (2) and (3), and therefore the effective theory, are the temperature T , the colour conductivity σ , and the self coupling of the gauge field hidden in the definition of the covariant derivative $\mathbf{D}^{ab} = \delta^{ab} \nabla - g f^{abc} \mathbf{A}^c$.

The procedure of reformulating a LANGEVIN equation like Eq. (2) in the form of a field theoretic path integral is well-known [15–17]: According to Eq. (2), the gauge field evolves, starting from certain initial conditions, under the influence of the stochastic force. An arbitrary observable of the theory then is defined by some functional of the gauge field $F[\mathbf{A}]$ and given by the expectation value of that functional with respect to the possible realisations

of the stochastic force

$$\langle F[\mathbf{A}] \rangle = \int \mathcal{D}\zeta F[\mathbf{A}^s[\zeta]] \varrho[\zeta] = \int \mathcal{D}\zeta F[\mathbf{A}^s[\zeta]] \exp \left\{ -\frac{1}{4\sigma T} \int dt d^{D-1}x \zeta^a(t, \mathbf{x}) \cdot \zeta^a(t, \mathbf{x}) \right\} \quad (4)$$

Here we have denoted by $\mathbf{A}^s[\zeta]$ the solution of Eq. (2) for a specific choice of the stochastic force and the given initial conditions.

To proceed and recast the effective theory of the gauge field into a form resembling the path integral formulation of an ‘ordinary’ quantum field theory we would rather like to have a path integral running over the gauge field than running over the stochastic force. This can be achieved by inserting unity in an appropriate way. In fact, one has

$$1 = \int \mathcal{D}\mathbf{E} \delta(\mathbf{E} - \zeta) = \int_{\text{(i.c.)}} \mathcal{D}\mathbf{A} \text{Det} \left(\frac{\delta \mathbf{E}[\mathbf{A}]}{\delta \mathbf{A}} \right) \delta(\mathbf{E}[\mathbf{A}] - \zeta) \quad (5)$$

where we choose the functional $\mathbf{E}[\mathbf{A}]$ as the left-hand side of Eq. (2)

$$\mathbf{E}^a[\mathbf{A}] = \mathbf{D}^{ab} \times \mathbf{B}^b + \sigma \dot{\mathbf{A}}^a \quad (6)$$

The invertibility of $\mathbf{E}[\mathbf{A}]$ is essential to justify the change of variables in Eq. (5). It follows from the parabolic nature of the expression and from the restriction to those gauge field configurations in the second path integral satisfying the initial conditions.

Because Eq. (5) holds independently of ζ , it can be inserted into the path integral (4). The delta function then assures that only those gauge field configurations contribute to the integral that obey $\mathbf{E}[\mathbf{A}] = \zeta$. Due to our choice of $\mathbf{E}[\mathbf{A}]$, however, this is identical to the condition $\mathbf{A} = \mathbf{A}^s[\zeta]$. Thus, after inserting the delta function we may replace $\mathbf{A}^s[\zeta]$ in the path integral simply by the integration variable \mathbf{A} , and we are left with

$$\langle F[\mathbf{A}] \rangle = \int \mathcal{D}\zeta \varrho[\zeta] \int_{\text{(i.c.)}} \mathcal{D}\mathbf{A} \text{Det} \left(\frac{\delta \mathbf{E}[\mathbf{A}]}{\delta \mathbf{A}} \right) \delta(\mathbf{E}[\mathbf{A}] - \zeta) F[\mathbf{A}] \quad (7)$$

Moreover, the restriction to field configurations obeying a specific set of initial conditions can be dropped if these initial conditions are specified at $t = -\infty$. This is a consequence of their transversal component always being damped and the fact that any longitudinal contribution drops out whenever a gauge invariant observable is calculated. In case of a gauge variant quantity, however, a damping of the longitudinal component can be achieved by introducing an additional gauge fixing term into the LANGEVIN equation [16]. This will be necessary anyway in the following section in order to generalise from $A_0 = 0$ gauge. Henceforth, we will therefore drop the restriction on the path integration in Eq. (7).

At this point, one has two choices. One possibility is to proceed by doing the ζ integral with the help of the delta function. This results in a theory containing only the gauge field (and perhaps some additional ghost fields to be introduced later), however, at the expense of rather complicated interactions: the functional $\mathbf{E}[\mathbf{A}]$ shows up as argument of the gaussian probability distribution, and since $\mathbf{E}[\mathbf{A}]$ contains terms up to \mathbf{A}^3 , the action would inherit vertices of up to sixth order.

To avoid this situation, we instead choose to introduce an additional auxiliary field $\boldsymbol{\lambda}$ to represent the delta function

$$\delta(\mathbf{E}[\mathbf{A}] - \zeta) = \int \mathcal{D}\boldsymbol{\lambda} \exp \left\{ i \int dt d^{D-1}x \boldsymbol{\lambda}^a \cdot (\mathbf{E}^a[\mathbf{A}] - \zeta^a) \right\} \quad (8)$$

In this way, one can still perform the ζ integral that becomes gaussian, thereby eliminating the stochastic force field from the theory. One obtains

$$\langle F[\mathbf{A}] \rangle = \int \mathcal{D}\mathbf{A} \mathcal{D}\boldsymbol{\lambda} \text{Det} \left(\frac{\delta \mathbf{E}[\mathbf{A}]}{\delta \mathbf{A}} \right) F[\mathbf{A}] e^{-S[\mathbf{A}, \boldsymbol{\lambda}]} \quad (9)$$

with

$$S[\mathbf{A}, \boldsymbol{\lambda}] = \int dx \left[\sigma T \boldsymbol{\lambda}^a \cdot \boldsymbol{\lambda}^a - i \boldsymbol{\lambda}^a \cdot \mathbf{E}^a[\mathbf{A}] \right] \quad (10)$$

The determinant in Eq. (9) need not be taken into account since it can be shown to be a constant in dimensional regularisation (see Appendix A for an explicit calculation). We could, nevertheless, introduce a ghost representation of the determinant referring to the corresponding ghost fields as equation of motion (EOM) ghosts in the following. As a benefit of doing so the action (10) would be endowed with a BRST symmetry, allowing to easily obtain a kind of Ward identities (so-called stochastic Ward identities) reflecting the origin of the theory in a stochastic differential equation. Since it is desirable to obtain as many non-perturbative identities as possible in order to find a judicious ansatz for the truncation of the DYSON–SCHWINGER equations, introducing EOM ghosts, at first, seems the natural way to proceed.

However there is another type of Ward identities related to gauge invariance. Unfortunately, the gauge ghosts to be introduced to obtain these gauge Ward identities will break the stochastic BRST symmetries. So, instead of introducing EOM ghosts now, we will later introduce gauge ghosts in order to obtain the gauge Ward identities. The stochastic Ward identities will be derived without the help of a BRST symmetry by directly referring to

the fundamental structure of the theory that reflects its origin in a stochastic differential equation.

For now, absorbing the constant determinant in the measure, we are left with

$$\langle F[\mathbf{A}] \rangle = \int \mathcal{D}\mathbf{A} \mathcal{D}\boldsymbol{\lambda} F[\mathbf{A}] e^{-S[\mathbf{A}, \boldsymbol{\lambda}]} \quad (11)$$

where the action S is given by Eq. (10).

B. Upgrading to κ Gauge

Bödeker's theory is written in $A_0 = 0$ gauge, and so is our transcription as field theoretic path integral so far. At the end of the day, however, we will be forced to use an approximation to solve the non-perturbative equations obtained, e.g. DYSON–SCHWINGER equations, and this approximation might introduce gauge artefacts into the calculation. In order to allow some control over the gauge dependence of the results, we need to base our derivations on a reformulation of Bödeker's equation in a more general gauge.

In [12], Zinn-Justin and Zwanziger have shown that adding a term to Eq. (2) that is tangent to the gauge orbit

$$\mathbf{D}^{ab} \times \mathbf{B}^b + \sigma(\dot{\mathbf{A}}^a + \mathbf{D}^{ab} v^b[\mathbf{A}]) = \boldsymbol{\zeta}^a \quad (12)$$

has no effect on expectation values of gauge-invariant objects of the form $F[\mathbf{A}]$. This is not the most general modification of Eq. (2) which leaves expectation values of gauge invariant objects unchanged [14], but it suffices for our purposes. As long as $v^a[\mathbf{A}]$ contains no time derivatives, the added term has no effect in calculations of gauge invariant objects.

We can reformulate this fact in a different way: Since the non-abelian electric field is given by $\mathbf{E}^a = -\dot{\mathbf{A}}^a - \mathbf{D}^{ab} A^{b0}$, one may rewrite Bödeker's equation in the compact form

$$\mathbf{D}^{ab} \times \mathbf{B}^b - \sigma \mathbf{E}^a = \boldsymbol{\zeta}^a \quad (13)$$

which then may be interpreted in any of the so-called flow gauges $A^{a0} = v^a[\mathbf{A}]$ with no time derivatives allowed inside the functional $v^a[\mathbf{A}]$.

The restriction that $v^a[\mathbf{A}]$ does not contain time derivatives plays a more substantial role in our context than in the context of stochastic quantisation which was the object of Zinn-Justin and Zwanziger: In stochastic quantisation the time variable describes a *fictitious* time

that is introduced only as a device to reinterpret a given Euclidean quantum field theory as the limit of a stochastic process for large values of the fictitious time [16]. Absence of time derivatives in stochastic quantisation therefore means absence of derivatives with respect to fictitious time and does not pose any restrictions to usual time derivatives. In our context, on the contrary, time is the real, physical time and the restrictions above narrow down the class of possible gauges leading to a well defined LANGEVIN equation.

Moreover, because of the different role of the time variable, we also have a component of the gauge field that is associated with the t variable of the LANGEVIN equation. In stochastic quantisation this is not the case because t is fictitious and the time associated with A_0 is just the zero component of the Euclidean \mathbf{x} vector. To cope with this different structure, to some extent will demand a generalisation of the proof of Zinn-Justin and Zwanziger.

In effect, we not only have to prove that gauge invariant objects of the form $F[\mathbf{A}]$ are left invariant by the introduction of the term $v^a[\mathbf{A}]$, as was shown in [12]. Instead we have to prove the following: Given Bödeker's equation in the form (13) and a gauge invariant functional $F[A^0, \mathbf{A}]$, then any choice of a flow gauge leads to the same result. Or put in different words, calculating $\langle F[v[\mathbf{A}], \mathbf{A}] \rangle$ by means of the equation Eq. (12) gives always the same value, independent of $v[\mathbf{A}]$.

We now proceed in a similar manner to [12]. Let us consider the left-hand side of Eq. (12) where we add a small variation of the $v^a[\mathbf{A}]$ term. We evaluate this expression for a gauge field that is subject to an arbitrary, infinitesimal gauge transformation $\mathbf{A}'^a = \mathbf{A}^a + \mathbf{D}^{ab}\omega^b$ and find

$$\begin{aligned} & \mathbf{D}'^{ab} \times \mathbf{B}'^b + \sigma(\dot{\mathbf{A}}'^a + \mathbf{D}'^{ab}v^b[\mathbf{A}'] + \mathbf{D}'^{ab}\delta v^b[\mathbf{A}']) \\ &= (\delta^{ab} + g f^{abc}\omega^c) \left[\mathbf{D}^{bd} \times \mathbf{B}^d + \sigma(\dot{\mathbf{A}}^b + \mathbf{D}^{bd}v^d[\mathbf{A}]) \right] + \sigma \mathbf{D}^{ab} \left[\frac{\partial \omega^b}{\partial t} + [H[\mathbf{A}]\omega]^b + \delta v^b[\mathbf{A}] \right] \end{aligned} \quad (14)$$

Here we have used

$$\mathbf{D}'^{ab} \times \mathbf{B}'^b = (\delta^{ab} + g f^{abc}\omega^c) \mathbf{D}^{bd} \times \mathbf{B}^d \quad (15)$$

$$\dot{\mathbf{A}}'^a = (\delta^{ab} + g f^{abc}\omega^c) \dot{\mathbf{A}}^b + \mathbf{D}^{ab} \frac{\partial \omega^b}{\partial t} \quad (16)$$

i.e. the product $\mathbf{D}^{ab} \times \mathbf{B}^b$ transforms covariantly whereas the transformation of $\dot{\mathbf{A}}^a$ has a covariant and non-covariant contribution. In the same way we have split the transformation of $v^a[\mathbf{A}]$ into a covariant and non-covariant part: Starting from

$$v^a[\mathbf{A}'](t, \mathbf{x}) = v^a[\mathbf{A}](t, \mathbf{x}) + \int d^{D-1}y \frac{\delta v^a[\mathbf{A}](t, \mathbf{x})}{\delta A^{bi}(t, \mathbf{y})} \delta A^{bi}(t, \mathbf{y}) \quad (17)$$

we have indeed

$$v^a[\mathbf{A}'](t, \mathbf{x}) = (\delta^{ab} + g f^{abc} \omega^c) v^b[\mathbf{A}](t, \mathbf{x}) + [H[\mathbf{A}]\omega]^a(t, \mathbf{x}) \quad (18)$$

where $\delta A^{bi} = D_i^{bc} \omega^c$ has been used and we have introduced the abbreviation

$$[H[\mathbf{A}]\omega]^a(t, \mathbf{x}) = \int d^{D-1}y \frac{\delta v^a[\mathbf{A}](t, \mathbf{x})}{\delta A^{bi}(t, \mathbf{y})} (D_i^{bc} \omega^c)(t, \mathbf{y}) - g f^{abc} v^b[\mathbf{A}](t, \mathbf{x}) \omega^c(t, \mathbf{x}) \quad (19)$$

Note that the functional derivatives in Eqs. (17) and (19) are only with respect to a spacial variation because $v^a[\mathbf{A}]$ does not contain any time derivatives (otherwise we would also have to integrate over time). Let us give the explicit form of this somewhat frightening expression for $H[\mathbf{A}]\omega$ in the case of the choice $v^a[\mathbf{A}] = -\frac{1}{\kappa} \nabla \cdot \mathbf{A}^a$. One simply obtains

$$[H[\mathbf{A}]\omega]^a(t, \mathbf{x}) = -\frac{1}{\kappa} (\mathbf{D}^{ab} \cdot \nabla \omega^b)(t, \mathbf{x}) \quad (20)$$

Finally, Eq. (18) leads to

$$\mathbf{D}'^{ab} v^b[\mathbf{A}'] = (\delta^{ab} + g f^{abc} \omega^c) \mathbf{D}^{bd} v^d[\mathbf{A}] + \mathbf{D}^{ab} [H[\mathbf{A}]\omega]^b \quad (21)$$

where it was used that ω is infinitesimal and of course

$$\mathbf{D}'^{ab} \delta v^b[\mathbf{A}'] = \mathbf{D}^{ab} \delta v^b[\mathbf{A}] \quad (22)$$

because δv is infinitesimal itself.

Let us now come back to Eq. (14) and its meaning. Suppose the gauge field, before the gauge transformation has been performed, was a solution of Bödeker's equation with the $v^a[\mathbf{A}]$ term present, but without the additional $\delta v^a[\mathbf{A}]$ term. In other words, the original gauge field was a solution of Eq. (12). We can then replace the first square bracket on the right-hand side of Eq. (14) by the stochastic force and find

$$\begin{aligned} & \mathbf{D}'^{ab} \times \mathbf{B}'^b + \sigma (\dot{\mathbf{A}}'^a + \mathbf{D}'^{ab} v^b[\mathbf{A}'] + \mathbf{D}'^{ab} \delta v^b[\mathbf{A}']) \\ &= \zeta'^a + \sigma \mathbf{D}^{ab} \left[\frac{\partial \omega^b}{\partial t} + [H[\mathbf{A}]\omega]^b + \delta v^b[\mathbf{A}] \right] \end{aligned} \quad (23)$$

This means, if we subject the original gauge field to an arbitrary, infinitesimal gauge transformation with parameter ω , then the gauge transformed field will be a solution of Eq. (23), i.e. of the original equation with v replaced by $v + \delta v$ and the stochastic force transformed in the same way as the gauge field ... but with an ugly additional term on the right-hand

side. However, one can play a dirty trick: What was said so far was true for an *arbitrary* gauge transformation. But if we demand ω to be a solution of

$$\frac{\partial \omega^b}{\partial t} + [H[\mathbf{A}]\omega]^b + \delta v^b[\mathbf{A}] = 0 \quad (24)$$

then the square bracket on the right of Eq. (23) will vanish and we finally arrive at

$$\mathbf{D}'^{ab} \times \mathbf{B}'^b + \sigma(\dot{\mathbf{A}}'^a + \mathbf{D}'^{ab} v^b[\mathbf{A}'] + \mathbf{D}'^{ab} \delta v^b[\mathbf{A}']) = \zeta'^a \quad (25)$$

However, there is a certain subtlety that we want to draw attention to. To clarify this point, let us once again repeat the line of reasoning: Starting with a gauge field being solution of

$$\mathbf{D}^{ab} \times \mathbf{B}^b + \sigma(\dot{\mathbf{A}}^a + \mathbf{D}^{ab} v^b[\mathbf{A}]) = \zeta^a \quad (26)$$

we search for a gauge transformation ω that obeys

$$\frac{\partial \omega^a}{\partial t} + [H[\mathbf{A}]\omega]^a + \delta v^a[\mathbf{A}] = 0 \quad (27)$$

(and we can always find such an ω because (27) is a linear, inhomogeneous equation with given inhomogeneity $\delta v^a[\mathbf{A}]$). Then the gauge field transformed with *this* ω , $\mathbf{A}'^a = \mathbf{A}^a + \mathbf{D}^{ab} \omega^b$, is a solution of the original equation with v replaced by $v + \delta v$ and the stochastic force also transformed by the same ω

$$\mathbf{D}'^{ab} \times \mathbf{B}'^b + \sigma(\dot{\mathbf{A}}'^a + \mathbf{D}'^{ab} v^b[\mathbf{A}'] + \mathbf{D}'^{ab} \delta v^b[\mathbf{A}']) = \zeta'^a \quad (28)$$

The subtle point is the following: The original gauge field \mathbf{A} is a solution of Eq. (26) and thus depends on the stochastic force ζ , of course. But \mathbf{A} is an input of Eq. (27) that determines ω . Therefore, ω via \mathbf{A} too depends on ζ . As a consequence of this, ζ' inherits a non-trivial dependence on ζ : The stochastic force ζ' not only depends on ζ because it is the gauge transform of ζ , but also because the gauge transformation itself depends on ζ

$$\zeta'^a = (\delta^{ab} + g f^{abc} \omega^c[\zeta]) \zeta^b \quad (29)$$

We denote by $\mathbf{A}^s[\zeta, v, \mathbf{A}_{\text{ini}}]$ the solution of Eq. (12) for the specific realisation ζ of the stochastic force term and initial conditions \mathbf{A}_{ini} . Correspondingly, let $\mathbf{A}^s[\zeta, v + \delta v, \mathbf{A}_{\text{ini}}]$ denote the solution of this equation with v replaced by $v + \delta v$ and for the same stochastic force and initial conditions. We can then express the contents of Eq. (28) in this new notation

$$\mathbf{A}^s[\zeta, v + \delta v, \mathbf{A}_{\text{ini}}] = {}^\omega \mathbf{A}^s[\zeta, v, \mathbf{A}_{\text{ini}}] \quad (30)$$

where the superscript ω indicates gauge transformation with the special parameter ω corresponding to the solution on the right-hand side via Eq. (27).

After these preparations we can now show that gauge invariant expectation values $\langle F[A^0, \mathbf{A}] \rangle$ are independent of the choice of $v^a[\mathbf{A}]$. To this end, let us write the gauge invariant observable as functional of the non-abelian electric and magnetic field

$$\begin{aligned} \mathbf{E}^a &= -\dot{\mathbf{A}}^a - \mathbf{D}^{ab} A^{b0} \\ \mathbf{B}^a &= \nabla \times \mathbf{A}^a + \frac{1}{2} g f^{abc} \mathbf{A}^b \times \mathbf{A}^c \end{aligned} \quad (31)$$

We then have

$$\langle F[\mathbf{E}, \mathbf{B}] \rangle_{v+\delta v} = \int \mathcal{D}\zeta' \varrho[\zeta'] F[\mathbf{E}_{v+\delta v}[\mathbf{A}], \mathbf{B}_{v+\delta v}[\mathbf{A}]]_{\mathbf{A}=\mathbf{A}^s[\zeta', v+\delta v, \mathbf{A}'_{\text{ini}}]} \quad (32)$$

with

$$\mathbf{E}_{v+\delta v}^a[\mathbf{A}] = -\dot{\mathbf{A}}^a - \mathbf{D}^{ab} v^b[\mathbf{A}] - \mathbf{D}^{ab} \delta v^b[\mathbf{A}] \quad (33)$$

and $\mathbf{B}_{v+\delta v}[\mathbf{A}] = \mathbf{B}_v[\mathbf{A}]$ as in Eq. (31). Changing variables according to Eq. (29), one obtains

$$\langle F[\mathbf{E}, \mathbf{B}] \rangle_{v+\delta v} = \int \mathcal{D}\zeta \text{Det} \left(\frac{\delta \omega \zeta}{\delta \zeta} \right) \varrho[\omega \zeta] F[\mathbf{E}_{v+\delta v}[\mathbf{A}], \mathbf{B}_{v+\delta v}[\mathbf{A}]]_{\mathbf{A}=\mathbf{A}^s[\omega \zeta, v+\delta v, \mathbf{A}'_{\text{ini}}]} \quad (34)$$

We now use independence on the initial conditions, the transformation property (30), gauge invariance of $\varrho[\zeta]$ and finally the fact that the determinant is unity (shown in Appendix A). This all together leads to

$$\langle F[\mathbf{E}, \mathbf{B}] \rangle_{v+\delta v} = \int \mathcal{D}\zeta \varrho[\zeta] F[\mathbf{E}_{v+\delta v}[\omega \mathbf{A}], \mathbf{B}_{v+\delta v}[\omega \mathbf{A}]]_{\mathbf{A}=\mathbf{A}^s[\zeta, v, \mathbf{A}_{\text{ini}}]} \quad (35)$$

Taking into account the transformation properties (16), (21) and (22), we find

$$\begin{aligned} \mathbf{E}_{v+\delta v}^a[\omega \mathbf{A}] &= (\omega \mathbf{E}_v[\mathbf{A}])^a - \mathbf{D}^{ab} \left[\frac{\partial \omega^b}{\partial t} + [H[\mathbf{A}] \omega]^b + \delta v^b[\mathbf{A}] \right] = (\omega \mathbf{E}_v[\mathbf{A}])^a \\ \mathbf{B}_{v+\delta v}^a[\omega \mathbf{A}] &= (\omega \mathbf{B}_v[\mathbf{A}])^a \end{aligned} \quad (36)$$

and thus

$$\langle F[\mathbf{E}, \mathbf{B}] \rangle_{v+\delta v} = \int \mathcal{D}\zeta \varrho[\zeta] F[\omega \mathbf{E}_v[\mathbf{A}], \omega \mathbf{B}_v[\mathbf{A}]]_{\mathbf{A}=\mathbf{A}^s[\zeta, v, \mathbf{A}_{\text{ini}}]} = \langle F[\mathbf{E}, \mathbf{B}] \rangle_v \quad (37)$$

because $F[\mathbf{E}, \mathbf{B}]$ is a gauge invariant functional.

Consequently, we have shown that Bödeker's equation in $A_0 = 0$ gauge

$$\mathbf{D}^{ab} \times \mathbf{B}^b + \sigma \dot{\mathbf{A}}^a = \zeta^a \quad (38)$$

can equivalently be formulated in any flow gauge

$$\mathbf{D}^{ab} \times \mathbf{B}^b + \sigma(\dot{\mathbf{A}}^a + \mathbf{D}^{ab}v^b[\mathbf{A}]) = \boldsymbol{\zeta}^a \quad (39)$$

without any time derivatives allowed inside the functional $v^a[\mathbf{A}]$. We will henceforth use the special choice $A^{a0} = v^a[\mathbf{A}] = -\frac{1}{\kappa} \nabla \cdot \mathbf{A}^a$ and refer to it as κ gauge. This is a natural choice for $v^a[\mathbf{A}]$, since it has the lowest order in \mathbf{A} , preserves colour invariance, and with $\kappa > 0$ the term $\mathbf{D}^{ab}v^b[\mathbf{A}]$ provides a globally restoring force along gauge orbits [13], while at the same time having the correct dimensions.

III. BRST SYMMETRIC ACTION AND WARD-TAKAHASHI IDENTITIES

We have argued that in order to derive any reliable statements from our theory, it is essential to gain some control over the gauge dependence possibly introduced by the truncation of the DYSON–SCHWINGER equations. This was our main motivation to generalise Bödeker’s equation from $A_0 = 0$ gauge to a more general class of flow gauges. In addition to this, the corresponding introduction of a gauge-fixing force has a welcome side-effect: It solves at the same time the problem of undamped longitudinal components of the initial gauge field configuration.

However, the detection of an unphysical gauge dependence is not what we really want; in fact, we would rather like to avoid it. The ultimate goal is to construct a truncation scheme that is physically reasonable and does not (or, realistically speaking, only slightly) violate the gauge symmetry.

To this end, we need identities expressing the gauge symmetry on the level of n-point functions, i.e. we need the WARD–TAKAHASHI identities of the theory.¹

Any physically reasonable truncation will have to respect these identities. Besides this conceptual importance, we may also hope that some of the Ward identities to be derived in the following will be of some practical use in solving the DYSON–SCHWINGER equations: In ordinary QCD, for instance, the full gluon propagator in covariant gauge is restricted to

¹ In the non-abelian context, these identities are often referred to as SLAVNOV–TAYLOR identities. However, following the terminology of Ref. [12], we denote these identities as gauge Ward identities in analogy to the stochastic Ward identities also encountered in this work.

being purely transversal as a consequence of the Ward identities. This leads, of course, to a great simplification in the DYSON–SCHWINGER equations of QCD.

In this section, we study three different kinds of non-perturbative identities: gauge Ward identities, i.e. SLAVNOV–TAYLOR identities; stochastic Ward identities; and ghost number conservation.

A. Constructing a BRST Symmetric Action

In Section II B, we saw that Eq. (12) transforms covariantly only under a restricted class of gauge transformations. Obtaining the gauge Ward identities with this restriction turns out to be rather cumbersome. Instead, we will raise to life the gauge parameter ω by introducing into the theory an additional (Grassmann valued) field that realises the constraint on the gauge transformations. The resulting action will be endowed with a BRST symmetry, and we will be able to obtain the gauge Ward identities in a straight-forward manner.

Setting $\delta v^a[\mathbf{A}]$ to zero in Eq. (24) we see that Eq. (12) transforms covariantly under gauge transformations which obey

$$\frac{\partial \omega^b}{\partial t} + [H[\mathbf{A}]\omega]^b = 0 \quad (40)$$

Note that the introduction of $v^a[\mathbf{A}]$ does not restrict the gauge group any further than it already would be. Even without the extra term, the gauge transformations would have to be restricted in order for Eq. (12) to be gauge covariant.

The restriction in Eq. (40) can be taken into account in the path integral in the following way. Define a term $\gamma^a[\omega, \mathbf{A}]$ from the left-hand side of Eq. (40), which for our choice of the functional $v^a[\mathbf{A}]$ takes the form

$$\gamma^a[\omega, \mathbf{A}] = \frac{\partial \omega^a}{\partial t} - \frac{1}{\kappa} \mathbf{D}^{ab} \cdot \nabla \omega^b \quad (41)$$

Perform a change of variables from γ to ω in the following Grassmann integral representation of unity

$$1 = \int \mathcal{D}\gamma \delta(\gamma) = \int \mathcal{D}\omega \frac{1}{\text{Det}\left(\frac{\delta \gamma[\omega, \mathbf{A}]}{\delta \omega}\right)} \delta\left(\frac{\partial \omega^a}{\partial t} - \frac{1}{\kappa} \mathbf{D}^{ab} \cdot \nabla \omega^b\right) \quad (42)$$

Since the determinant is Grassmann even, it no longer depends on ω and it can be pulled out of the integral. The determinant is a constant, and can be calculated in a similar manner to the determinant in Eq. (9); see Appendix A for an explicit calculation.

Inserting the integral representation of the Grassmann delta function

$$\delta(\gamma) = \int \mathcal{D}\bar{\omega} \exp \left\{ \int dx \bar{\omega}^a(x) \gamma^a(x) \right\} \quad (43)$$

and absorbing the constant determinant into the measure, we find the identity

$$1 = \int \mathcal{D}\omega \mathcal{D}\bar{\omega} \exp \left\{ \int dx \bar{\omega}^a(x) \left(\delta^{ab} \frac{\partial}{\partial t} - \frac{1}{\kappa} \mathbf{D}^{ab} \cdot \nabla \right) \omega^b(x) \right\} \quad (44)$$

which holds independently of the gauge field \mathbf{A} . Therefore, it can be inserted into the analogous of the path integral representation of the generating functional, Eq. (11), based on the generalised version of Bödeker's equation (12). This leads to

$$Z[\mathbf{J}] = \int \mathcal{D}\mathbf{A} \mathcal{D}\boldsymbol{\lambda} \mathcal{D}\omega \mathcal{D}\bar{\omega} \exp \left\{ -S[\mathbf{A}, \boldsymbol{\lambda}, \omega, \bar{\omega}] + \int dx \mathbf{J}^a(x) \mathbf{A}^a(x) \right\} \quad (45)$$

with the action now given by

$$S[\mathbf{A}, \boldsymbol{\lambda}, \omega, \bar{\omega}] = S^{(\text{D})}[\mathbf{A}, \boldsymbol{\lambda}] + S^{(\text{GG})}[\mathbf{A}, \omega, \bar{\omega}] \quad (46)$$

where $S^{(\text{D})}[\mathbf{A}, \boldsymbol{\lambda}]$ is the generalised contribution of the dynamical fields

$$S^{(\text{D})}[\mathbf{A}, \boldsymbol{\lambda}] = \int dx \left[\sigma T \boldsymbol{\lambda}^a \cdot \boldsymbol{\lambda}^a - i \boldsymbol{\lambda}^a \cdot \left(\mathbf{D}^{ab} \times \mathbf{B}^b + \sigma (\dot{\mathbf{A}}^a - \frac{1}{\kappa} \mathbf{D}^{ab} \nabla \cdot \mathbf{A}^b) \right) \right] \quad (47)$$

and

$$S^{(\text{GG})}[\mathbf{A}, \omega, \bar{\omega}] = \int dx \left[-\bar{\omega}^a \dot{\omega}^a + \frac{1}{\kappa} \bar{\omega}^a \mathbf{D}^{ab} \cdot \nabla \omega^b \right] \quad (48)$$

is the new contribution containing the gauge ghosts ω and $\bar{\omega}$.

B. Gauge Ward Identities

The SLAVNOV–TAYLOR identities can be derived by noting that the action (46) is invariant under the following BRST transformation

$$\begin{aligned} \delta_\varepsilon \mathbf{A}^a(x) &= \mathbf{D}^{ab}(x) \varepsilon \omega^b(x) & \delta_\varepsilon \omega^a(x) &= \frac{1}{2} g f^{abc} \varepsilon \omega^c(x) \omega^b(x) \\ \delta_\varepsilon \boldsymbol{\lambda}^a(x) &= g f^{abc} \varepsilon \omega^c(x) \boldsymbol{\lambda}^b(x) & \delta_\varepsilon \bar{\omega}^a(x) &= g f^{abc} \varepsilon \omega^c(x) \bar{\omega}^b(x) + i \varepsilon \sigma \mathbf{D}^{ab}(x) \cdot \boldsymbol{\lambda}^b(x) \end{aligned} \quad (49)$$

where ε is a constant Grassmann parameter. It is convenient to introduce the finite BRST operator s such that the result of s acting on a functional of the fields \mathbf{A} , $\boldsymbol{\lambda}$, ω and $\bar{\omega}$ is defined as (left) derivative with respect to the parameter ε of the variations in Eq. (49). We thus have

$$sF[\mathbf{A}, \boldsymbol{\lambda}, \omega, \bar{\omega}] = \frac{\partial}{\partial \varepsilon} \delta_\varepsilon F[\mathbf{A}, \boldsymbol{\lambda}, \omega, \bar{\omega}] \quad (50)$$

or conversely

$$\delta_\varepsilon F[\mathbf{A}, \boldsymbol{\lambda}, \omega, \bar{\omega}] = \varepsilon s F[\mathbf{A}, \boldsymbol{\lambda}, \omega, \bar{\omega}] \quad (51)$$

From Eq. (50) one finds the following representation

$$s = \int dx \left[(sA^{ai}) \frac{\delta}{\delta A^{ai}} + (s\lambda^{ai}) \frac{\delta}{\delta \lambda^{ai}} + (s\omega^a) \frac{\delta}{\delta \omega^a} + (s\bar{\omega}^a) \frac{\delta}{\delta \bar{\omega}^a} \right] \quad (52)$$

with the finite BRST transforms of the fundamental fields given by Eq. (49)

$$\begin{aligned} s\mathbf{A}^a(x) &= \mathbf{D}^{ab}(x) \omega^b(x) & s\omega^a(x) &= \frac{1}{2} g f^{abc} \omega^c(x) \omega^b(x) \\ s\boldsymbol{\lambda}^a(x) &= g f^{abc} \omega^c(x) \boldsymbol{\lambda}^b(x) & s\bar{\omega}^a(x) &= g f^{abc} \omega^c(x) \bar{\omega}^b(x) + i\sigma \mathbf{D}^{ab}(x) \cdot \boldsymbol{\lambda}^b(x) \end{aligned} \quad (53)$$

The BRST operator s has two essential properties: it annihilates the complete action (46)

$$sS[\mathbf{A}, \boldsymbol{\lambda}, \omega, \bar{\omega}] = 0 \quad (54)$$

expressing the invariance of $S[\mathbf{A}, \boldsymbol{\lambda}, \omega, \bar{\omega}]$ under the BRST transformation (49), and it's nilpotency

$$s^2 = 0 \quad (55)$$

Using the operator s , we now define the generating functional in the following way

$$\begin{aligned} Z[J, I] &= \int \mathcal{D}\mathbf{A} \mathcal{D}\boldsymbol{\lambda} \mathcal{D}\omega \mathcal{D}\bar{\omega} \exp \left\{ -S[\mathbf{A}, \boldsymbol{\lambda}, \omega, \bar{\omega}] + \int dx \left[\mathbf{A}^a \cdot \mathbf{J}_A^a + \boldsymbol{\lambda}^a \cdot \mathbf{J}_\lambda^a + \omega^a J_\omega^a + \bar{\omega}^a J_{\bar{\omega}}^a \right. \right. \\ &\quad \left. \left. + \mathbf{I}_{sA}^a \cdot s\mathbf{A}^a + \mathbf{I}_{s\lambda}^a \cdot s\boldsymbol{\lambda}^a + I_{s\omega}^a s\omega^a + I_{s\bar{\omega}}^a s\bar{\omega}^a \right] \right\} \end{aligned} \quad (56)$$

Note that $\omega, \bar{\omega}, s\mathbf{A}$ and $s\boldsymbol{\lambda}$ together with their sources $J_\omega, J_{\bar{\omega}}, \mathbf{I}_{sA}, \mathbf{I}_{s\lambda}$ are Grassmann odd, the remaining quantities Grassmann even.

We proceed to vary the fields in Eq. (56) according to Eq. (49). The Jacobian of such a transformation is unity due to Eq. (53) as can be seen from the explicit calculation in Appendix A. We also know that the action is invariant under this change of variables $S[\mathbf{A}, \boldsymbol{\lambda}, \omega, \bar{\omega}] = S[\mathbf{A}', \boldsymbol{\lambda}', \omega', \bar{\omega}']$. In addition, the source terms of the BRST transformed fields are invariant due to the nilpotency of s and the fact that the variations are s -transforms themselves, e.g. $\delta_\varepsilon \mathbf{A}' = \varepsilon s\mathbf{A}'$. Only the source terms of the fundamental fields are not invariant and transform according to

$$\mathbf{A}^a \cdot \mathbf{J}_A^a = \mathbf{A}'^a \cdot \mathbf{J}_A^a + \delta_\varepsilon \mathbf{A}'^a \cdot \mathbf{J}_A^a = \mathbf{A}'^a \cdot \mathbf{J}_A^a + \varepsilon s\mathbf{A}'^a \cdot \mathbf{J}_A^a \quad (57)$$

and likewise for the other fields. Thus, under the change of variables (49), the integrand in Eq. (56) is simply reproduced with all fields replaced by their primed counterparts and an additional factor

$$\exp\left\{\varepsilon \int dx \left[s\mathbf{A}'^a \cdot \mathbf{J}_A^a + s\boldsymbol{\lambda}'^a \cdot \mathbf{J}_\lambda^a + s\omega'^a J_\omega^a + s\bar{\omega}'^a J_{\bar{\omega}}^a \right] \right\} \quad (58)$$

generated by the transformation of the fundamental source terms, Eq. (57). Because ε is Grassmann odd we have

$$\exp\left\{\varepsilon \int dx \left[s\mathbf{A}^a \cdot \mathbf{J}_A^a + s\boldsymbol{\lambda}^a \cdot \mathbf{J}_\lambda^a + s\omega^a J_\omega^a + s\bar{\omega}^a J_{\bar{\omega}}^a \right] \right\} = 1 + \varepsilon \int dx \left[s\mathbf{A}'^a \cdot \mathbf{J}_A^a + s\boldsymbol{\lambda}'^a \cdot \mathbf{J}_\lambda^a + s\omega'^a J_\omega^a + s\bar{\omega}'^a J_{\bar{\omega}}^a \right]$$

Inserted back into the path integral Eq. (56), the one just gives $Z[J, I]$, which cancels the left-hand side of the equation. Hence, we obtain

$$0 = \int \mathcal{D}\mathbf{A} \mathcal{D}\boldsymbol{\lambda} \mathcal{D}\omega \mathcal{D}\bar{\omega} \varepsilon \int dx \left[s\mathbf{A}^a \cdot \mathbf{J}_A^a + s\boldsymbol{\lambda}^a \cdot \mathbf{J}_\lambda^a + s\omega^a J_\omega^a + s\bar{\omega}^a J_{\bar{\omega}}^a \right] \exp\left\{(\dots)\right\} \quad (59)$$

where the dots represent the exponential in Eq. (56). This has to be true for any ε and thus the expression without ε has to vanish itself. Changing the BRST transformed fields for functional derivatives with respect to their sources, we find the following identity

$$\int dx \left[J_A^{ai}(x) \frac{\delta}{\delta I_{sA}^{ai}(x)} + J_\lambda^{ai}(x) \frac{\delta}{\delta I_{s\lambda}^{ai}(x)} + J_\omega^a(x) \frac{\delta}{\delta I_{s\omega}^a(x)} + J_{\bar{\omega}}^a(x) \frac{\delta}{\delta I_{s\bar{\omega}}^a(x)} \right] Z[J, I] = 0 \quad (60)$$

Finally, let us transcribe this relation in an identity for the generating functional of one-particle irreducible (1PI) correlation functions. To this end, we first express it by the generating functional of connected correlation functions $W[J, I] = \ln Z[J, I]$. In terms of $W[J, I]$ the relation (60) reads

$$\int dx \left[J_A^{ai}(x) \frac{\delta W[J, I]}{\delta I_{sA}^{ai}(x)} + J_\lambda^{ai}(x) \frac{\delta W[J, I]}{\delta I_{s\lambda}^{ai}(x)} + J_\omega^a(x) \frac{\delta W[J, I]}{\delta I_{s\omega}^a(x)} + J_{\bar{\omega}}^a(x) \frac{\delta W[J, I]}{\delta I_{s\bar{\omega}}^a(x)} \right] = 0 \quad (61)$$

To define the generating functional of one-particle irreducible correlation functions, we introduce the usual expectation values for the fields in the presence of the external sources

$$\begin{aligned} A^{ai}(x) &= \frac{\delta W[J, I]}{\delta J_A^{ai}(x)} & \omega^a(x) &= -\frac{\delta W[J, I]}{\delta J_\omega^a(x)} \\ \lambda^{ai}(x) &= \frac{\delta W[J, I]}{\delta J_\lambda^{ai}(x)} & \bar{\omega}^a(x) &= -\frac{\delta W[J, I]}{\delta J_{\bar{\omega}}^a(x)} \end{aligned} \quad (62)$$

The minus signs in the case of the ghost fields are a consequence of our definition of the generating functional, Eq. (56), where we ordered the sources to the right of the fundamental fields.

Assuming that the relations (62) can be solved for the sources J , we can define the 1PI generating functional Γ as the LEGENDRE transform of $W[J, I]$ with respect to the sources J . The sources of the BRST transformed fields are not LEGENDRE transformed and play the role of spectators only. With the definition

$$\Gamma[\mathbf{A}, \boldsymbol{\lambda}, \omega, \bar{\omega}; I] = \int dx [\mathbf{A}^a \cdot \mathbf{J}_A^a + \boldsymbol{\lambda}^a \cdot \mathbf{J}_\lambda^a + \omega^a J_\omega^a + \bar{\omega}^a J_{\bar{\omega}}^a] - W[J, I] \quad (63)$$

one finds

$$\begin{aligned} \frac{\delta \Gamma}{\delta A^{ai}(x)} &= J_A^{ai}(x) & \frac{\delta \Gamma}{\delta \omega^a(x)} &= J_\omega^a(x) \\ \frac{\delta \Gamma}{\delta \lambda^{ai}(x)} &= J_\lambda^{ai}(x) & \frac{\delta \Gamma}{\delta \bar{\omega}^a(x)} &= J_{\bar{\omega}}^a(x) \end{aligned} \quad (64)$$

and also

$$\begin{aligned} \frac{\delta \Gamma}{\delta I_{sA}^{ai}(x)} &= -\frac{\delta W}{\delta I_{sA}^{ai}(x)} & \frac{\delta \Gamma}{\delta I_{s\omega}^a(x)} &= -\frac{\delta W}{\delta I_{s\omega}^a(x)} \\ \frac{\delta \Gamma}{\delta I_{s\lambda}^{ai}(x)} &= -\frac{\delta W}{\delta I_{s\lambda}^{ai}(x)} & \frac{\delta \Gamma}{\delta I_{s\bar{\omega}}^a(x)} &= -\frac{\delta W}{\delta I_{s\bar{\omega}}^a(x)} \end{aligned} \quad (65)$$

which may be used to reexpress the gauge Ward identity (61) in terms of Γ

$$\int dx \left[\frac{\delta \Gamma}{\delta A^{ai}(x)} \frac{\delta \Gamma}{\delta I_{sA}^{ai}(x)} + \frac{\delta \Gamma}{\delta \lambda^{ai}(x)} \frac{\delta \Gamma}{\delta I_{s\lambda}^{ai}(x)} + \frac{\delta \Gamma}{\delta \omega^a(x)} \frac{\delta \Gamma}{\delta I_{s\omega}^a(x)} + \frac{\delta \Gamma}{\delta \bar{\omega}^a(x)} \frac{\delta \Gamma}{\delta I_{s\bar{\omega}}^a(x)} \right] = 0 \quad (66)$$

C. Stochastic Ward Identities

We have included in Eq. (56) the auxiliary field $\boldsymbol{\lambda}$ and the ghost fields ω and $\bar{\omega}$, all of which were not strictly necessary, but rather were included so as to facilitate our work. They could, in principle, be integrated out and we would be left with Eq. (11), except that we have now also introduced sources for the extra fields, as well as for the BRST transformed ones. This would suggest that there could be some sort of relations for the generating functional in Eq. (56) resulting from our choice to include the extra fields and sources.

To derive these relations for Bödeker's effective theory, one starts from the generating functional (56), including sources of the fundamental as well as the (gauge) BRST transformed fields. Inserting the action and the BRST transforms according to Eqs. (46) – (48) and Eq. (53) with the definitions (6) and (41) in use, the generating functional $Z[J, I]$ may

be written

$$\begin{aligned}
Z[J, I] = & \int \mathcal{D}\mathbf{A} \mathcal{D}\boldsymbol{\lambda} \mathcal{D}\omega \mathcal{D}\bar{\omega} \exp \left\{ \int dx \left[-\sigma T \boldsymbol{\lambda}^a \cdot \boldsymbol{\lambda}^a + i \boldsymbol{\lambda}^a \cdot (\mathbf{E}^a[\mathbf{A}] - i \mathbf{J}_\lambda^a + ig f^{abc} \omega^b \mathbf{I}_{s\lambda}^c - \sigma \mathbf{D}^{ab} I_{s\bar{\omega}}^b) \right. \right. \\
& + \bar{\omega}^a (\gamma^a[\omega, \mathbf{A}] + J_{\bar{\omega}}^a - gf^{abc} \omega^b I_{s\bar{\omega}}^c) + \mathbf{A}^a \cdot \mathbf{J}_A^a + \omega^a J_\omega^a \\
& \left. \left. + \mathbf{I}_{sA}^a \cdot \mathbf{D}^{ab} \omega^b + I_{s\omega}^a \frac{1}{2} g f^{abc} \omega^c \omega^b \right] \right\} \tag{67}
\end{aligned}$$

where terms multiplying $\boldsymbol{\lambda}$ and $\bar{\omega}$ have been collected. Because the exponent is quadratic in the former and linear in the latter, both of these fields can be integrated. One obtains

$$\begin{aligned}
Z[J, I] = & \int \mathcal{D}\mathbf{A} \mathcal{D}\omega \delta(\gamma') \exp \left\{ \int dx \left[-\frac{1}{4\sigma T} \mathbf{E}'^a \cdot \mathbf{E}'^a + \mathbf{A}^a \cdot \mathbf{J}_A^a + \omega^a J_\omega^a \right. \right. \\
& \left. \left. + \mathbf{I}_{sA}^a \cdot \mathbf{D}^{ab} \omega^b + I_{s\omega}^a \frac{1}{2} g f^{abc} \omega^c \omega^b \right] \right\} \tag{68}
\end{aligned}$$

with the new functionals \mathbf{E}' and γ' defined as

$$\mathbf{E}'^a[\omega, \mathbf{A}; \mathbf{J}_\lambda, \mathbf{I}_{s\lambda}, I_{s\bar{\omega}}] = \mathbf{E}^a[\mathbf{A}] - i \mathbf{J}_\lambda^a + ig f^{abc} \omega^b \mathbf{I}_{s\lambda}^c - \sigma \mathbf{D}^{ab} I_{s\bar{\omega}}^b \tag{69}$$

$$\gamma'^a[\omega, \mathbf{A}; J_{\bar{\omega}}, I_{s\bar{\omega}}] = \gamma^a[\omega, \mathbf{A}] + J_{\bar{\omega}}^a - gf^{abc} \omega^b I_{s\bar{\omega}}^c \tag{70}$$

Hence, when restricting to vanishing sources $\mathbf{J}_A = \mathbf{I}_{sA} = 0$ and $J_\omega = I_{s\omega} = 0$ the exponent becomes purely quadratic in \mathbf{E}' . Defining for brevity

$$Z_1[\mathbf{J}_\lambda, J_{\bar{\omega}}, \mathbf{I}_{s\lambda}, I_{s\bar{\omega}}] = Z[\mathbf{J}_A = 0, \mathbf{J}_\lambda, J_\omega = 0, J_{\bar{\omega}}, \mathbf{I}_{sA} = 0, \mathbf{I}_{s\lambda}, I_{s\omega} = 0, I_{s\bar{\omega}}] \tag{71}$$

we have

$$Z_1[\mathbf{J}_\lambda, J_{\bar{\omega}}, \mathbf{I}_{s\lambda}, I_{s\bar{\omega}}] = \int \mathcal{D}\mathbf{A} \mathcal{D}\omega \delta(\gamma') \exp \left\{ -\frac{1}{4\sigma T} \int dx \mathbf{E}'^a \cdot \mathbf{E}'^a \right\} \tag{72}$$

where \mathbf{E}' and γ' both depend on \mathbf{A} and ω as indicated in Eqs. (69) and (70). Thus, it is quite natural to attempt a change of variables from \mathbf{A} and ω to \mathbf{E}' and γ' . The Jacobian can be calculated in a similar manner as the Jacobian of Eq. (42), and again can be shown to be a constant. The resulting integral is gaussian and evaluates to a constant functional Z_1 leading to

$$Z_1[\mathbf{J}_\lambda, J_{\bar{\omega}}, \mathbf{I}_{s\lambda}, I_{s\bar{\omega}}] = \text{const.} \tag{73}$$

or likewise for $W_1 = \ln Z_1$

$$W_1[\mathbf{J}_\lambda, J_{\bar{\omega}}, \mathbf{I}_{s\lambda}, I_{s\bar{\omega}}] = \text{const.} \tag{74}$$

As a consequence, any combination of functional derivatives with respect to sources chosen from the class $\{\mathbf{J}_\lambda, J_{\bar{\omega}}, \mathbf{I}_{s\lambda}, I_{s\bar{\omega}}\}$ yields zero when acting on the full generating functionals and evaluated for vanishing sources:

$$\underbrace{\frac{\delta}{\delta \dots} \frac{\delta}{\delta \dots} \dots \frac{\delta}{\delta \dots}}_{\text{any combination}} W[J, I] \Big|_{J=I=0} = 0 \quad (75)$$

of $J_\lambda, J_{\bar{\omega}}, I_{s\lambda}, I_{s\bar{\omega}}$

with the same relation holding for derivatives of $Z[J, I]$. To obtain a corresponding identity for the 1PI generating functional Γ , note that due to Eq. (74) one has on the submanifold defined by the vanishing of the four sources $\mathbf{J}_A, \mathbf{I}_{sA}, J_\omega$ and $I_{s\omega}$

$$\lambda^{ai}(x) \Big|_{\substack{\mathbf{J}_A = \mathbf{I}_{sA} = 0 \\ J_\omega = I_{s\omega} = 0}} = \frac{\delta W_1}{\delta J_\lambda^{ai}(x)} = 0 \quad \text{and} \quad \bar{\omega}^a(x) \Big|_{\substack{\mathbf{J}_A = \mathbf{I}_{sA} = 0 \\ J_\omega = I_{s\omega} = 0}} = -\frac{\delta W_1}{\delta J_{\bar{\omega}}^a(x)} = 0 \quad (76)$$

So Γ could at most depend on \mathbf{A} , ω , $\mathbf{I}_{s\lambda}$ and $I_{s\bar{\omega}}$. However, from Eq. (64) we have

$$\frac{\delta \Gamma}{\delta A^{ai}(x)} = J_A^{ai}(x) \quad \frac{\delta \Gamma}{\delta \omega^a(x)} = J_\omega^a(x) \quad (77)$$

and therefore Γ may not depend on \mathbf{A} or ω anymore. The same conclusion can be reached for the $\mathbf{I}_{s\lambda}$ and $I_{s\bar{\omega}}$ by looking at Eq. (65) and Eq. (75). Therefore, Γ must be a constant, this leads to

$$\underbrace{\frac{\delta}{\delta \dots} \frac{\delta}{\delta \dots} \dots \frac{\delta}{\delta \dots}}_{\text{any combination}} \Gamma[\mathbf{A}, \boldsymbol{\lambda}, \omega, \bar{\omega}; I] \Big|_{J=I=0} = 0 \quad (78)$$

of $A, \omega, I_{s\lambda}, I_{s\bar{\omega}}$

which is the equivalent of the stochastic Ward identity (75) in terms of the 1PI generating functional Γ .

D. Ghost Number Conservation

We will discuss one last symmetry of the action (46). The action is invariant under the global transformation

$$\begin{aligned} \omega^a(x) &= e^{i\alpha} \omega'^a(x) \\ \bar{\omega}^a(x) &= e^{-i\alpha} \bar{\omega}'^a(x) \end{aligned} \quad (79)$$

of the ghost and anti-ghost fields. In addition to this, subjecting the measure $\mathcal{D}\omega\mathcal{D}\bar{\omega}$ to the transformation (79), i.e. to

$$\begin{aligned} & (\omega^a(x_1), \bar{\omega}^a(x_1), \omega^a(x_2), \bar{\omega}^a(x_2), \dots) \\ &= (e^{i\alpha}\omega'^a(x_1), e^{-i\alpha}\bar{\omega}'^a(x_1), e^{i\alpha}\omega'^a(x_2), e^{-i\alpha}\bar{\omega}'^a(x_2), \dots) \end{aligned} \quad (80)$$

one finds

$$\mathcal{D}\omega\mathcal{D}\bar{\omega} = \prod_{a,n} [d\omega^a(x_n) d\bar{\omega}^a(x_n)] = \prod_{a,n} [d\omega'^a(x_n) d\bar{\omega}'^a(x_n)] J(\omega', \bar{\omega}') \quad (81)$$

with the Jacobian

$$J^{-1}(\omega', \bar{\omega}') = \det \begin{pmatrix} e^{+i\alpha} & 0 & 0 & 0 & \dots \\ 0 & e^{-i\alpha} & 0 & 0 & \dots \\ 0 & 0 & e^{+i\alpha} & 0 & \dots \\ 0 & 0 & 0 & e^{-i\alpha} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = 1 \quad (82)$$

Hence, the measure is also invariant under the transformation (79)

$$\mathcal{D}\omega\mathcal{D}\bar{\omega} = \mathcal{D}\omega'\mathcal{D}\bar{\omega}' \quad (83)$$

Together with the invariance of the action, this symmetry leads to ghost number conservation, which poses another restriction on the form of the generating functionals and their derivatives. Indeed, taking the parameter α in Eq. (79) to be infinitesimal and performing the corresponding change of variables

$$\begin{aligned} \omega^a(x) &= \omega'^a(x) + i\alpha\omega'^a(x) \\ \bar{\omega}^a(x) &= \bar{\omega}'^a(x) - i\alpha\bar{\omega}'^a(x) \end{aligned} \quad (84)$$

in the defining path integral (56) of the generating functional $Z[J, I]$ yields

$$Z[J, I] = \int \mathcal{D}\mathbf{A} \mathcal{D}\boldsymbol{\lambda} \mathcal{D}\omega \mathcal{D}\bar{\omega} \exp \left\{ i\alpha \int dx \left[\omega^a J_\omega^a - \bar{\omega}^a J_{\bar{\omega}}^a + \mathbf{I}_{sA}^a \cdot s\mathbf{A}^a + \mathbf{I}_{s\bar{\omega}}^a \cdot s\boldsymbol{\lambda}^a + 2I_{s\omega}^a s\omega^a \right] \right\} \exp \left\{ (\dots) \right\}$$

Here we have already renamed the primed symbols again to unprimed ones after the change of variables has been completed. As before, the dots represent the original exponent as it occurs in Eq. (56). Using the fact that α is assumed to be infinitesimal, we can expand the first exponential and replace any fields that appear by functional derivatives acting on

the exponential (after interchanging the order of the ghost and anti-ghost field and their corresponding sources leading to a minus sign in either case). The derivatives can finally be pulled out of the functional integral and we obtain

$$\int dx \left[J_\omega^a(x) \frac{\delta}{\delta J_\omega^a(x)} - J_{\bar{\omega}}^a(x) \frac{\delta}{\delta J_{\bar{\omega}}^a(x)} + I_{sA}^{ai}(x) \frac{\delta}{\delta I_{sA}^{ai}(x)} + I_{s\lambda}^{ai}(x) \frac{\delta}{\delta I_{s\lambda}^{ai}(x)} + 2I_{s\omega}^a(x) \frac{\delta}{\delta I_{s\omega}^a(x)} \right] Z[J, I] = 0 \quad (85)$$

Again, the definition $W[J, I] = \ln Z[J, I]$ implies that the same identity holds for the generating functional $W[J, I]$ of connected correlation functions

$$\int dx \left[J_\omega^a \frac{\delta W}{\delta J_\omega^a} - J_{\bar{\omega}}^a \frac{\delta W}{\delta J_{\bar{\omega}}^a} + I_{sA}^{ai} \frac{\delta W}{\delta I_{sA}^{ai}} + I_{s\lambda}^{ai} \frac{\delta W}{\delta I_{s\lambda}^{ai}} + 2I_{s\omega}^a \frac{\delta W}{\delta I_{s\omega}^a} \right] = 0 \quad (86)$$

where we have suppressed the space-time argument x and the dependence of W on the sources J and I . This identity in turn can easily be translated to the corresponding restriction on the 1PI generating functional $\Gamma[\mathbf{A}, \boldsymbol{\lambda}, \omega, \bar{\omega}; I]$. By means of Eqs. (62), (64) and (65) one finds

$$\int dx \left[\frac{\delta \Gamma}{\delta \omega^a} \omega^a - \frac{\delta \Gamma}{\delta \bar{\omega}^a} \bar{\omega}^a + I_{sA}^{ai} \frac{\delta \Gamma}{\delta I_{sA}^{ai}} + I_{s\lambda}^{ai} \frac{\delta \Gamma}{\delta I_{s\lambda}^{ai}} + 2I_{s\omega}^a \frac{\delta \Gamma}{\delta I_{s\omega}^a} \right] = 0 \quad (87)$$

This concludes our derivation of non-perturbative identities for the generating functional (56). Explicit forms of these identities for lower N-point functions are shown in Appendix C.

IV. DYSON–SCHWINGER EQUATIONS

To derive the DYSON–SCHWINGER equations, we observe that the path integral of a functional derivative vanishes, i.e.

$$\int \mathcal{D}\phi \frac{\delta}{\delta \phi(x)} F[\phi] = 0 \quad (88)$$

for any functional $F[\phi]$. Hence, in the case of Bödeker’s theory, we obtain four different equations by inserting a functional derivative with respect to each of the fields \mathbf{A} , $\boldsymbol{\lambda}$, ω or $\bar{\omega}$ into the generating functional

$$\begin{aligned} Z[J, I] = \int \mathcal{D}\mathbf{A} \mathcal{D}\boldsymbol{\lambda} \mathcal{D}\omega \mathcal{D}\bar{\omega} \exp \left\{ -S[\mathbf{A}, \boldsymbol{\lambda}, \omega, \bar{\omega}] + \int dx \left[\mathbf{A}^a \cdot \mathbf{J}_A^a + \boldsymbol{\lambda}^a \cdot \mathbf{J}_\lambda^a + \omega^a J_\omega^a + \bar{\omega}^a J_{\bar{\omega}}^a \right. \right. \\ \left. \left. + \mathbf{I}_{sA}^a \cdot s\mathbf{A}^a + \mathbf{I}_{s\lambda}^a \cdot s\boldsymbol{\lambda}^a + I_{s\omega}^a s\omega^a + I_{s\bar{\omega}}^a s\bar{\omega}^a \right] \right\} \end{aligned} \quad (89)$$

A. General Dyson–Schwinger Equations

1. Ghost (ω) and Anti-ghost ($\bar{\omega}$) equations

Starting from the identity

$$0 = \int \mathcal{D}\mathbf{A} \mathcal{D}\boldsymbol{\lambda} \mathcal{D}\omega \mathcal{D}\bar{\omega} \frac{\delta}{\delta \omega^a(x)} \exp\{(\dots)\} \quad (90)$$

where the dots represent the exponent of Eq. (89), gives

$$\begin{aligned} 0 = & \int \mathcal{D}\mathbf{A} \mathcal{D}\boldsymbol{\lambda} \mathcal{D}\omega \mathcal{D}\bar{\omega} \left[\dot{\bar{\omega}}^a(x) + \frac{1}{\kappa} \mathbf{D}^{ab}(x) \cdot \nabla \bar{\omega}^b(x) - \frac{g}{\kappa} f^{abc} \bar{\omega}^b(x) \nabla \cdot \mathbf{A}^c(x) + J_\omega^a(x) + \nabla \cdot \mathbf{I}_{sA}^a(x) \right. \\ & \left. + g f^{abc} (-\mathbf{I}_{sA}^b(x) \cdot \mathbf{A}^c(x) - \mathbf{I}_{s\lambda}^b(x) \cdot \boldsymbol{\lambda}^c(x) + I_{s\omega}^b(x) \omega^c(x) + I_{s\bar{\omega}}^b(x) \bar{\omega}^c(x)) \right] \exp\{(\dots)\} \end{aligned} \quad (91)$$

Expressing the fields by derivatives acting on the exponential, one obtains

$$\begin{aligned} Z J_\omega^a(x) = & (\partial_t + \frac{1}{\kappa} \Delta) \frac{\delta Z}{\delta J_\omega^a(x)} - \frac{g}{\kappa} f^{abc} \partial_j \frac{\delta^2 Z}{\delta J_\omega^b(x) \delta J_A^{cj}(x)} - Z \nabla \cdot \mathbf{I}_{sA}^a(x) \\ & + g f^{abc} \left[I_{sA}^{bj}(x) \frac{\delta Z}{\delta J_A^{cj}(x)} + I_{s\lambda}^{bj}(x) \frac{\delta Z}{\delta J_\lambda^{cj}(x)} + I_{s\omega}^b(x) \frac{\delta Z}{\delta J_\omega^c(x)} + I_{s\bar{\omega}}^b(x) \frac{\delta Z}{\delta J_{\bar{\omega}}^c(x)} \right] \end{aligned} \quad (92)$$

or in terms of W

$$\begin{aligned} J_\omega^a(x) = & (\partial_t + \frac{1}{\kappa} \Delta) \frac{\delta W}{\delta J_\omega^a(x)} - \frac{g}{\kappa} f^{abc} \partial_j \left[\frac{\delta^2 W}{\delta J_\omega^b(x) \delta J_A^{cj}(x)} + \frac{\delta W}{\delta J_\omega^b(x)} \frac{\delta W}{\delta J_A^{cj}(x)} \right] - \nabla \cdot \mathbf{I}_{sA}^a(x) \\ & + g f^{abc} \left[I_{sA}^{bj}(x) \frac{\delta W}{\delta J_A^{cj}(x)} + I_{s\lambda}^{bj}(x) \frac{\delta W}{\delta J_\lambda^{cj}(x)} + I_{s\omega}^b(x) \frac{\delta W}{\delta J_\omega^c(x)} + I_{s\bar{\omega}}^b(x) \frac{\delta W}{\delta J_{\bar{\omega}}^c(x)} \right] \end{aligned} \quad (93)$$

Transcription to the 1PI generating functional Γ yields

$$\begin{aligned} \frac{\delta \Gamma}{\delta \omega^a(x)} = & -(\partial_t + \frac{1}{\kappa} \Delta) \bar{\omega}^a(x) - \frac{g}{\kappa} f^{abc} \partial_j \left[\frac{\delta^2 W}{\delta J_\omega^b(x) \delta J_A^{cj}(x)} - \bar{\omega}^b(x) A^{cj}(x) \right] - \nabla \cdot \mathbf{I}_{sA}^a(x) \\ & + g f^{abc} \left[I_{sA}^{bj}(x) A^{cj}(x) + I_{s\lambda}^{bj}(x) \lambda^{cj}(x) - I_{s\omega}^b(x) \omega^c(x) - I_{s\bar{\omega}}^b(x) \bar{\omega}^c(x) \right] \end{aligned} \quad (94)$$

The antighost equation is obtained in a similar manner and reads

$$\frac{\delta \Gamma}{\delta \bar{\omega}^a(x)} = (-\partial_t + \frac{1}{\kappa} \Delta) \omega^a(x) + \frac{g}{\kappa} f^{abc} \partial_j \left[\frac{\delta^2 W}{\delta J_\omega^b(x) \delta J_A^{cj}(x')} - \omega^b(x) A^{cj}(x') \right] - g f^{abc} I_{s\bar{\omega}}^b(x) \omega^c(x) \quad (95)$$

where x' is set to x after the space-time derivative is carried out, i.e. the derivative acts on the argument of $J_\omega^b(x)$ only.

2. Auxiliary field (λ) equation

To deduce the auxiliary field equation from

$$0 = \int \mathcal{D}\mathbf{A} \mathcal{D}\lambda \mathcal{D}\omega \mathcal{D}\bar{\omega} \frac{\delta}{\delta \lambda^{ai}(x)} \exp\{(\dots)\} \quad (96)$$

we need, among other things, the functional derivative of the action $S[\mathbf{A}, \lambda, \omega, \bar{\omega}] = S^{(D)}[\mathbf{A}, \lambda] + S^{(GG)}[\mathbf{A}, \omega, \bar{\omega}]$. However, in the present case the corresponding expression becomes rather cumbersome.

As for the FEYNMAN rules in Appendix B, we want to use a symmetrised $\lambda \mathbf{A}^2$ and $\lambda \mathbf{A}^3$ vertex. The λ dependence of the action spreads out over the three contributions to the dynamical action $S^{(D)}[\mathbf{A}, \lambda] = S_0^{(D)}[\mathbf{A}, \lambda] + S_{\text{int},3}^{(D)}[\mathbf{A}, \lambda] + S_{\text{int},4}^{(D)}[\mathbf{A}, \lambda]$. The corresponding derivatives can be written in the form

$$\frac{\delta S_0^{(D)}[\mathbf{A}, \lambda]}{\delta \lambda^{ai}(x)} = 2\sigma T \lambda^{ai}(x) - i \left[\delta^{ij} (\sigma \partial_t - \Delta) + \left(1 - \frac{\sigma}{\kappa}\right) \partial_i \partial_j \right] A^{aj}(x) \quad (97)$$

$$\begin{aligned} \frac{\delta S_{\text{int},3}^{(D)}[\mathbf{A}, \lambda]}{\delta \lambda^{ai}(x)} &= \frac{1}{2!} (-ig) f^{abc} \left[\left(1 - \frac{\sigma}{\kappa}\right) \left[\delta^{ij} \partial'_k - \delta^{ik} \partial'_j \right] + 2 \left[\delta^{ij} \partial_k - \delta^{ik} \partial'_j \right] \right. \\ &\quad \left. + \left[\delta^{jk} \partial'_i - \delta^{kj} \partial'_i \right] \right] A^{bj}(x) A^{ck}(x') \Big|_{x'=x} \end{aligned} \quad (98)$$

$$\frac{\delta S_{\text{int},4}^{(D)}[\mathbf{A}, \lambda]}{\delta \lambda^{ai}(x)} = \frac{1}{3!} (-ig^2) V_{ijkl}^{abcd} A^{bj}(x) A^{ck}(x) A^{dl}(x) \quad (99)$$

where V_{ijkl}^{abcd} is defined in Eq. (B.24). One then obtains in terms of the 1PI generating functional

$$\begin{aligned} \frac{\delta \Gamma}{\delta \lambda^{ai}(x)} &= 2\sigma T \lambda^{ai}(x) - i \left[\delta^{ij} (\sigma \partial_t - \Delta) + \left(1 - \frac{\sigma}{\kappa}\right) \partial_i \partial_j \right] A^{aj}(x) \\ &\quad - \frac{ig}{2!} f^{abc} \left[\left(1 - \frac{\sigma}{\kappa}\right) \left[\delta^{ij} \partial'_k - \delta^{ik} \partial'_j \right] + 2 \left[\delta^{ij} \partial_k - \delta^{ik} \partial'_j \right] + \left[\delta^{jk} \partial'_i - \delta^{kj} \partial'_i \right] \right] \\ &\quad \times \left[\frac{\delta^2 W}{\delta J_A^{bj}(x) \delta J_A^{ck}(x')} + A^{bj}(x) A^{ck}(x') \right]_{x'=x} \\ &\quad - \frac{ig^2}{3!} V_{ijkl}^{abcd} \left[\frac{\delta^3 W}{\delta J_A^{bj}(x) \delta J_A^{ck}(x) \delta J_A^{dl}(x)} + 3 \frac{\delta^2 W}{\delta J_A^{bj}(x) \delta J_A^{ck}(x)} A^{dl}(x) + A^{bj}(x) A^{ck}(x) A^{dl}(x) \right] \\ &\quad - g f^{abc} \left[-I_{s\lambda}^{bi}(x) \omega^c(x) + i\sigma I_{s\bar{\omega}}^b(x) A^{ci}(x) \right] + i\sigma \partial_i I_{s\bar{\omega}}^a(x) \end{aligned} \quad (100)$$

3. Gauge field (\mathbf{A}) equation

Finally, coming to the gauge field equation

$$0 = \int \mathcal{D}\mathbf{A} \mathcal{D}\boldsymbol{\lambda} \mathcal{D}\omega \mathcal{D}\bar{\omega} \frac{\delta}{\delta A^{ai}(x)} \exp\{(\dots)\} \quad (101)$$

and using the derivatives

$$\frac{\delta S_0^{(\text{D})}[\mathbf{A}, \boldsymbol{\lambda}]}{\delta A^{ai}(x)} = -i [\delta^{ij}(-\sigma\partial_t - \Delta) + (1 - \frac{\sigma}{\kappa}) \partial_i \partial_j] \lambda^{aj}(x) \quad (102)$$

$$\begin{aligned} \frac{\delta S_{\text{int},3}^{(\text{D})}[\mathbf{A}, \boldsymbol{\lambda}]}{\delta A^{ai}(x)} &= -ig f^{abc} \left[-\left(1 - \frac{\sigma}{\kappa}\right) [\delta^{ij} \partial'_k + \delta^{jk} (\partial_i + \partial'_i)] + 2 [\delta^{jk} \partial'_i + \delta^{ij} (\partial_k + \partial'_k)] \right. \\ &\quad \left. - [\delta^{ik} \partial'_j + \delta^{ik} (\partial_j + \partial'_j)] \right] \lambda^{bj}(x) A^{ck}(x') \Big|_{x'=x} \end{aligned} \quad (103)$$

$$\frac{\delta S_{\text{int},4}^{(\text{D})}[\mathbf{A}, \boldsymbol{\lambda}]}{\delta A^{ai}(x)} = \frac{1}{2!} (-ig^2) V_{lijk}^{dabc} \lambda^{dl}(x) A^{bj}(x) A^{ck}(x) \quad (104)$$

where the symmetry of V_{ijkl}^{abcd} has been exploited, together with

$$\frac{\delta S^{(\text{GG})}[\mathbf{A}, \omega, \bar{\omega}]}{\delta A^{ai}(x)} = -\frac{g}{\kappa} f^{abc} \bar{\omega}^b(x) \partial_i \omega^c(x) \quad (105)$$

one arrives at

$$\begin{aligned} \frac{\delta \Gamma}{\delta A^{ai}(x)} &= -i [\delta^{ij}(-\sigma\partial_t - \Delta) + (1 - \frac{\sigma}{\kappa}) \partial_i \partial_j] \lambda^{aj}(x) - \frac{ig^2}{2!} V_{lijk}^{dabc} \left[\frac{\delta^3 W}{\delta J_\lambda^{dl}(x) \delta J_A^{bj}(x) \delta J_A^{ck}(x)} \right. \\ &\quad \left. + 2 \frac{\delta^2 W}{\delta J_\lambda^{dl}(x) \delta J_A^{bj}(x)} A^{ck}(x) + \lambda^{dl}(x) \frac{\delta^2 W}{\delta J_A^{bj}(x) \delta J_A^{ck}(x)} + \lambda^{dl}(x) A^{bj}(x) A^{ck}(x) \right] \\ &\quad - ig f^{abc} \left[-\left(1 - \frac{\sigma}{\kappa}\right) [\delta^{ij} \partial'_k + \delta^{jk} (\partial_i + \partial'_i)] + 2 [\delta^{jk} \partial'_i + \delta^{ij} (\partial_k + \partial'_k)] \right. \\ &\quad \left. - [\delta^{ik} \partial'_j + \delta^{ik} (\partial_j + \partial'_j)] \right] \left[\frac{\delta^2 W}{\delta J_\lambda^{bj}(x) \delta J_A^{ck}(x')} + \lambda^{bj}(x) A^{ck}(x') \right]_{x'=x} \\ &\quad - \frac{g}{\kappa} f^{abc} \partial_i \left[\frac{\delta^2 W}{\delta J_{\bar{\omega}}^b(x') \delta J_{\omega}^c(x)} + \bar{\omega}^b(x') \omega^c(x) \right]_{x'=x} + g f^{abc} \left[I_{sA}^{bi}(x) \omega^c(x) + i\sigma I_{s\bar{\omega}}^b(x) \lambda^{ci}(x) \right] \end{aligned} \quad (106)$$

B. Explicit Equations for Lower N-Point Functions

1. Definitions and General Relations

Concerning the propagators, mixing will occur between the gauge field \mathbf{A} and the auxiliary field $\boldsymbol{\lambda}$, resulting in four possible propagators from the gauge/auxiliary field sector that can

be combined into one matrix propagator. These are completed by the propagator of the gauge ghosts. Altogether, we define the full (connected) propagators as

$$G^{(AA)ab}_{ij}(x, y) = \langle A^{ai}(x) A^{bj}(y) \rangle_c = \frac{\delta^2 W[J, I]}{\delta J_A^{ai}(x) \delta J_A^{bj}(y)} \Big|_{J=I=0} \quad (107)$$

$$G^{(\lambda A)ab}_{ij}(x, y) = \langle \lambda^{ai}(x) A^{bj}(y) \rangle_c = \frac{\delta^2 W[J, I]}{\delta J_\lambda^{ai}(x) \delta J_A^{bj}(y)} \Big|_{J=I=0} \quad (108)$$

$$G^{(\lambda\lambda)ab}_{ij}(x, y) = \langle \lambda^{ai}(x) \lambda^{bj}(y) \rangle_c = \frac{\delta^2 W[J, I]}{\delta J_\lambda^{ai}(x) \delta J_\lambda^{bj}(y)} \Big|_{J=I=0} \quad (109)$$

$$G^{(\omega)ab}(x, y) = \langle \omega^a(x) \bar{\omega}^b(y) \rangle_c = \frac{\delta^2 W[J, I]}{\delta J_\omega^a(x) \delta J_{\bar{\omega}}^b(y)} \Big|_{J=I=0} \quad (110)$$

and $G^{(A\lambda)ab}_{ij}(x, y) = G^{(\lambda A)ba}_{ji}(y, x)$ of course. In graphical representations we denote the gauge field by curly lines, the auxiliary field by double curly lines and the gauge ghosts by dotted lines. Thus, the full propagators are represented by

$$\begin{aligned} G^{(AA)ab}_{ij}(x, y) &= a, i \xrightarrow{x} \text{curly line} \xrightarrow{y} b, j \\ G^{(A\lambda)ab}_{ij}(x, y) &= a, i \xrightarrow{x} \text{curly line} \xrightarrow{y} b, j \\ G^{(\lambda A)ab}_{ij}(x, y) &= a, i \xrightarrow{x} \text{double curly line} \xrightarrow{y} b, j \\ G^{(\lambda\lambda)ab}_{ij}(x, y) &= a, i \xrightarrow{x} \text{double curly line} \xrightarrow{y} b, j \end{aligned}$$

and finally

$$G^{(\omega)ab}(x, y) = a \bullet \xleftarrow{x} \text{dotted line} \xleftarrow{y} b$$

Besides the propagators, we have to set out our definition for the self-energies. To this end, let us summarise the two left-hand equations of (64) in the form

$$J_F^{ai}(x) = \frac{\delta \Gamma[\mathbf{A}, \boldsymbol{\lambda}, \omega, \bar{\omega}; I]}{\delta F^{ai}(x)} \quad (111)$$

where the index F stands for any of the fields A or λ . Taking the functional derivative of this equation with respect to $J_G^{bj}(y)$, where again $G \in \{A, \lambda\}$, then yields (observing that \mathbf{A} , $\boldsymbol{\lambda}$, ω and $\bar{\omega}$ are functionals of the sources J and I)

$$\begin{aligned} \delta^{ab} \delta^{ij} \delta_{FG} \delta(x - y) &= \int dz \left[\frac{\delta A^{ck}(z)}{\delta J_G^{bj}(y)} \frac{\delta^2 \Gamma}{\delta A^{ck}(z) \delta F^{ai}(x)} + \frac{\delta \lambda^{ck}(z)}{\delta J_G^{bj}(y)} \frac{\delta^2 \Gamma}{\delta \lambda^{ck}(z) \delta F^{ai}(x)} \right. \\ &\quad \left. + \frac{\delta \omega^c(z)}{\delta J_G^{bj}(y)} \frac{\delta^2 \Gamma}{\delta \omega^c(z) \delta F^{ai}(x)} + \frac{\delta \bar{\omega}^c(z)}{\delta J_G^{bj}(y)} \frac{\delta^2 \Gamma}{\delta \bar{\omega}^c(z) \delta F^{ai}(x)} \right] \end{aligned} \quad (112)$$

Thus, using the Eqs. (62) to express the first factor in each term as a second derivative of W and finally setting the sources to zero leads to

$$\delta^{ab} \delta^{ij} \delta_{FG} \delta(x - y) = \int dz \left[G^{(GA)bc}_{jk}(y, z) \frac{\delta^2 \Gamma}{\delta A^{ck}(z) \delta F^{ai}(x)} \Big|_{J=I=0} + G^{(G\lambda)bc}_{jk}(y, z) \frac{\delta^2 \Gamma}{\delta \lambda^{ck}(z) \delta F^{ai}(x)} \Big|_{J=I=0} \right] \quad (113)$$

Here, the definitions (107) – (109) have been used and the terms involving ghost and anti-ghost fields have vanished due to ghost number conservation.

In the following we will often encounter multiple derivatives of the generating functionals W and Γ evaluated for vanishing sources. Let us therefore introduce a shorthand notation where we indicate the fields with respect to which the derivatives are taken as superscripts. Possible LORENTZ or colour indices as well as space-time arguments appear in the order of the fields they belong to. For instance, we abbreviate

$$\Gamma^{(\lambda A \bar{\omega})abc}_{ij}(x, y, z) = \frac{\delta^3 \Gamma}{\delta \lambda^{ai}(x) \delta A^{bj}(y) \delta \bar{\omega}^c(z)} \Big|_{J=I=0} \quad (114)$$

In the case of W , we also use the *fields* as superscripts though the derivatives are taken with respect to the corresponding *sources*, of course.

In this new notation, Eq. (113) reads

$$\delta^{ab} \delta^{ij} \delta_{FG} \delta(x - y) = \int dz G^{(GH)bc}_{jk}(y, z) \Gamma^{(HF)ca}_{ki}(z, x) \quad (115)$$

where H is a summation index running over the fields A and λ . This equation expresses the fact that the matrix propagator of the gauge/auxiliary field sector

$$\hat{G}_{ij}^{ab}(x, y) = \begin{pmatrix} G^{(\lambda\lambda)ab}_{ij}(x, y) & G^{(\lambda A)ab}_{ij}(x, y) \\ G^{(A\lambda)ab}_{ij}(x, y) & G^{(AA)ab}_{ij}(x, y) \end{pmatrix} \quad (116)$$

is inverse to the matrix

$$\hat{\Gamma}_{ij}^{ab}(x, y) = \begin{pmatrix} \Gamma^{(\lambda\lambda)ab}_{ij}(x, y) & \Gamma^{(\lambda A)ab}_{ij}(x, y) \\ \Gamma^{(A\lambda)ab}_{ij}(x, y) & \Gamma^{(AA)ab}_{ij}(x, y) \end{pmatrix} \quad (117)$$

constructed of the second derivatives of Γ . Consequently, the self-energy $\hat{\Pi}_{ij}^{ab}(x, y)$ is determined via the relation

$$\Gamma^{(FG)ab}_{ij}(x, y) = (\Delta^{-1})^{(FG)ab}_{ij}(x, y) + \Pi^{(FG)ab}_{ij}(x, y) \quad (118)$$

where $(\Delta^{-1})^{(FG)ab}_{ij}(x, y)$ are the components of the inverse free propagator of perturbation theory (see Appendix B, Eqs. (B.6) – (B.9)), and where $F, G \in \{\lambda, A\}$ as before.

Analogously, taking the derivative with respect to $J_\omega^b(y)$ of

$$J_\omega^a(x) = \frac{\delta \Gamma[\mathbf{A}, \boldsymbol{\lambda}, \omega, \bar{\omega}; I]}{\delta \omega^a(x)} \quad (119)$$

and performing the same manipulations as described above leads to

$$\delta^{ab} \delta(x - y) = - \int dz G^{(\omega)bc}(y, z) \frac{\delta^2 \Gamma}{\delta \bar{\omega}^c(z) \delta \omega^a(x)} \Big|_{J=I=0} \quad (120)$$

Hence, we define the self-energy of the gauge ghosts via

$$\Gamma^{(\bar{\omega}\omega)ab}(x, y) = - [(\Delta^{-1})^{(\omega)ab}(x, y) + \Pi^{(\omega)ab}(x, y)] \quad (121)$$

with the free inverse propagator $(\Delta^{-1})^{(\omega)ab}(x, y)$ given in Eq. (B.18). In our graphical representations we denote self-energies and other one-particle irreducible quantities by open circles.

Though generally we are using three-vectors, in the FOURIER transformation we use four-vector notation

$$f(x) = \int \frac{d^D k}{(2\pi)^D} e^{-ikx} f(k) \quad (122)$$

with $-ikx = -ik_0 t + i \mathbf{k} \cdot \mathbf{x}$. The proper vertex functions in momentum space are basically given by the FOURIER transforms of the various functional derivatives of the 1PI generating functional Γ . However, due to translational invariance of the theory, all these FOURIER transforms contain a delta function expressing momentum conservation at the vertex. It is therefore convenient to pull these delta functions out of the definitions of the vertex functions. In this way, the latter become functions of one momentum variable less than indicated by the number of external legs. For instance, we define

$$(2\pi)^D \delta^D(k_1 + k_2 + k_3) \Gamma^{(\bar{\omega}\omega G)abc}_j(k_1, k_2) = \int dx dy dz e^{-ik_1 x - ik_2 y - ik_3 z} \Gamma^{(\bar{\omega}\omega G)abc}_j(x, y, z) \quad (123)$$

or equivalently

$$\Gamma^{(\bar{\omega}\omega G)abc}_j(x, y, z) = \int \frac{d^D k_1}{(2\pi)^D} \frac{d^D k_2}{(2\pi)^D} e^{-ik_1(z-x) - ik_2(z-y)} \Gamma^{(\bar{\omega}\omega G)abc}_j(k_1, k_2) \quad (124)$$

Here, the two arguments of the proper vertex function $\Gamma^{(\bar{\omega}\omega G)abc}_j(k_1, k_2)$ refer to the (incoming) momenta along the ghost lines leaving and entering the vertex in this order.

The choice of the $N - 1$ momenta that are used as arguments of a vertex with N external legs is, of course, arbitrary and thereby a source of possible confusion. We therefore explicitly list the definitions of the other relevant vertex functions used in this work

$$\Gamma^{(FGH)abc}_{ijk}(x, y, z) = - \int \frac{d^D k_2}{(2\pi)^D} \frac{d^D k_3}{(2\pi)^D} e^{-ik_2(x-y)-ik_3(x-z)} \Gamma^{(FGH)abc}_{ijk}(k_2, k_3) \quad (125)$$

with k_2 and k_3 denoting the incoming momenta along the G and H line respectively, and

$$\begin{aligned} \Gamma^{(FGHK)abcd}_{ijkl}(x, y, z, w) = & - \int \frac{d^D k_2}{(2\pi)^D} \frac{d^D k_3}{(2\pi)^D} \frac{d^D k_4}{(2\pi)^D} e^{-ik_2(x-y)-ik_3(x-z)} \\ & e^{-ik_4(x-w)} \Gamma^{(FGHK)abcd}_{ijkl}(k_2, k_3, k_4) \end{aligned} \quad (126)$$

with incoming momenta k_2, k_3, k_4 along the G, H and K line. Note the minus signs in the last two equations. The definitions above are chosen in such a way that they reduce at leading order to the corresponding vertices of the FEYNMAN rules, i.e.

$$\Gamma^{(\bar{\omega}\omega A)abc}_{j}(k_1, k_2) = \frac{ig}{\kappa} f^{abc} k_2^j + \dots \quad (127)$$

$$\Gamma^{(\lambda A A)abc}_{ijk}(k_2, k_3) = -g V^{abc}_{ijk}(\mathbf{k}_2, \mathbf{k}_3) + \dots \quad (128)$$

$$\Gamma^{(\lambda A A A)abcd}_{ijkl}(k_2, k_3, k_4) = ig^2 V^{abcd}_{ijkl} + \dots \quad (129)$$

2. DSE for $\Pi^{(\omega)}(k)$

Let us again start with the ghost equations, being much simpler than the equations for the gauge/auxiliary field sector. By taking the derivative of Eq. (95) with respect to $\omega^b(y)$, one finds evaluated for vanishing sources

$$\left. \frac{\delta^2 \Gamma}{\delta \omega^b(y) \delta \bar{\omega}^a(x)} \right|_{J=I=0} = \underbrace{\delta^{ab} \left(-\partial_t + \frac{1}{\kappa} \Delta \right) \delta(x-y)}_{(\Delta^{-1})^{(\omega)ab}(x, y)} + \frac{g}{\kappa} f^{ade} \partial_j \frac{\delta}{\delta \omega^b(y)} \left. \frac{\delta^2 W[J, I]}{\delta J_\omega^d(x) \delta J_A^{ej}(x')} \right|_{x'=x} \quad (130)$$

Comparing to the definition of the self-energy of the gauge ghosts in Eq. (121) then leads to the relation

$$\Pi^{(\omega)ab}(x, y) = \frac{g}{\kappa} f^{ade} \partial_j \frac{\delta}{\delta \omega^b(y)} \left. \frac{\delta^2 W[J, I]}{\delta J_\omega^d(x) \delta J_A^{ej}(x')} \right|_{x'=x} \quad (131)$$

for the gauge ghost self-energy. If we carry out the functional derivative with respect to $\omega^b(y)$, four terms arise because any of the sources $\mathbf{J}_A, \mathbf{J}_\lambda, J_\omega$ and $J_{\bar{\omega}}$ depends on ω . However, due

to ghost number conservation three of these terms vanish when the sources are set to zero and one is left with²

$$\Pi^{(\omega)ab}(x, y) = \frac{g}{\kappa} f^{ade} \partial_j \int dv \left[\frac{\delta^2 \Gamma}{\delta \omega^b(y) \delta \bar{\omega}^c(v)} \frac{\delta^3 W}{\delta J_{\bar{\omega}}^c(v) \delta J_{\omega}^d(x) \delta J_A^{ej}(x')} \right]_{x'=x} \quad (132)$$

$J = I = 0$

Finally, we express the connected three-point function by its 1PI counterpart

$$W^{(\bar{\omega}\omega F)abc}_j(x, y, z) = \int du du' du'' G^{(\omega)a'a}(u, x) G^{(\omega)bb'}(y, u') G^{(FG)cc'}_{jj'}(z, u'') \Gamma^{(\omega\bar{\omega}G)a'b'c'}_{j'}(u, u', u'') \quad (133)$$

where F represents one of the fields λ or A and G is a summation index taking these two values. The shorthand notation used here was introduced in Eq. (114). Note that the order of the ghost and anti-ghost fields in Eq. (133) is changed from $W^{(\bar{\omega}\omega F)}$ to $\Gamma^{(\omega\bar{\omega}G)}$ and that the (full) gauge ghost propagator is $G^{(\omega)ab}(x, y) = W^{(\omega\bar{\omega})ab}(x, y)$, as defined in Eq. (110).

Now, inserting relation (133) into Eq. (132), using the property (120) of the two-point functions and

$$\Gamma^{(\omega\bar{\omega}G)a'b'c'}_{j'}(u, u', u'') = -\Gamma^{(\omega\omega G)b'a'c'}_{j'}(u', u, u'') \quad (134)$$

yields the DYSON–SCHWINGER equation

$$\Pi^{(\omega)ab}(x, y) = - \int du' du'' G^{(AG)ee'}_{jj'}(x, u'') \frac{g}{\kappa} f^{ade} \partial_j G^{(\omega)dd'}(x, u') \Gamma^{(\bar{\omega}\omega G)d'be'}_{j'}(u', y, u'') \quad (135)$$

Using the definition for the momentum space proper vertex Eq.(123), we transform to momentum space

$$\Pi^{(\omega)ab}(k) = - \int \frac{d^D k'}{(2\pi)^D} \frac{ig}{\kappa} f^{ade} k'^j G^{(AG)ee'}_{jj'}(k - k') G^{(\omega)dd'}(k') \Gamma^{(\bar{\omega}\omega G)d'be'}_{j'}(-k', k) \quad (136)$$

The structure of the DYSON–SCHWINGER equation (136) is illustrated in Fig. 1. In Eq. (136) the field index G has a summation index taking the values $G = \lambda$ and $G = A$. In the graphical representation of Eq. (136) such a summation is symbolised by a solid line. This short-hand notation will become even more important in the other DYSON–SCHWINGER equations to follow. Thus, the right-hand side of Fig. 1 stands for two individual diagrams.

² It should be clear that x' is set to x only after the space-time derivative is carried out. In order to avoid an extensive use of brackets we decided to assume in this and similar cases some thoughtfulness on the part of the reader.

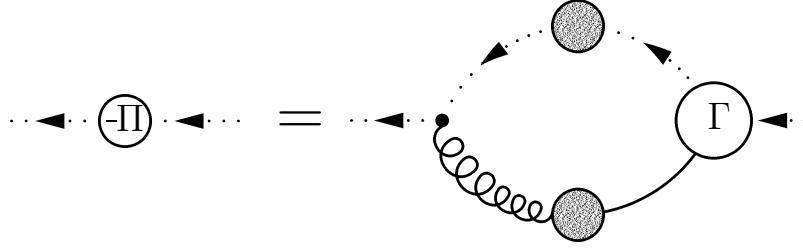


FIG. 1: DSE of the gauge ghost self-energy, Eqs. (136). Filled circles denote full propagators. Empty circles are used for one-particle irreducible quantities, i.e. self-energies and proper vertices. The solid line represents a summation of one graph with the line replaced by a gauge field and a second diagram with an auxiliary field instead.

Above we have deduced the DYSON–SCHWINGER equation of the gauge ghost self-energy from the general *anti-ghost* equation (95). A complementary relation can be obtained from the *ghost* equation (94). By taking the derivative with respect to $\bar{\omega}^b(y)$ of Eq. (94), one obtains

$$\Pi^{(\omega) ab}(k) = - \int \frac{d^D k'}{(2\pi)^D} \frac{ig}{\kappa} f^{dbe} k^j G^{(GA)e' e}(k - k') G^{(\omega) d' d}(k') \Gamma^{(\bar{\omega}\omega G)ad' e'}(-k, k') \quad (137)$$

3. DSE for $\Pi^{(\lambda\lambda)}(k)$

We come now to the DYSON–SCHWINGER equations of the gauge/auxiliary field sector. Taking the derivative with respect to $\lambda^{bj}(y)$ of the auxiliary field equation (100) yields after setting the sources to zero

$$\begin{aligned} \frac{\delta^2 \Gamma}{\delta \lambda^{ai}(x) \delta \lambda^{bj}(y)} \Big|_{J=I=0} &= \overbrace{2\sigma T \delta^{ab} \delta^{ij} \delta(x-y)} - \frac{ig^2}{3!} V_{iklm}^{acde} \frac{\delta}{\delta \lambda^{bj}(y)} \frac{\delta^3 W}{\delta J_A^{ck}(x) \delta J_A^{dl}(x) \delta J_A^{em}(x)} \Big|_{J=I=0} \\ &\quad - \frac{ig}{2!} f^{acd} \left[\left(1 - \frac{\sigma}{\kappa}\right) [\delta^{ik} \partial_l' - \delta^{il} \partial_k] + 2 [\delta^{ik} \partial_l - \delta^{il} \partial_k'] \right. \\ &\quad \left. + [\delta^{kl} \partial_i' - \delta^{lk} \partial_i] \right] \frac{\delta}{\delta \lambda^{bj}(y)} \frac{\delta^2 W}{\delta J_A^{ck}(x) \delta J_A^{dl}(x')} \Big|_{x'=x} \quad (138) \end{aligned}$$

Thus, comparing to Eq. (118) one reads off the self-energy component

$$\begin{aligned}
\Pi^{(\lambda\lambda)ab}_{ij}(x, y) = & -\frac{ig}{2!} f^{acd} \left[\left(1 - \frac{\sigma}{\kappa}\right) [\delta^{ik}\partial'_l - \delta^{il}\partial_k] + 2 [\delta^{ik}\partial_l - \delta^{il}\partial'_k] \right. \\
& \left. + [\delta^{kl}\partial'_i - \delta^{lk}\partial_i] \right] \frac{\delta}{\delta\lambda^{bj}(y)} \frac{\delta^2 W}{\delta J_A^{ck}(x) \delta J_A^{dl}(x')} \Big|_{x'=x} \\
& J=I=0 \\
& -\frac{ig^2}{3!} V_{iklm}^{acde} \frac{\delta}{\delta\lambda^{bj}(y)} \frac{\delta^3 W}{\delta J_A^{ck}(x) \delta J_A^{dl}(x) \delta J_A^{em}(x)} \Big|_{J=I=0} \tag{139}
\end{aligned}$$

To evaluate Eq. (139), we have to calculate the remaining functional derivatives and finally transform into momentum space. Let us start with the λ derivative of the connected two-point function. Because we will encounter similar expressions also in the DYSON–SCHWINGER equations of the other self-energy components, it is useful to generalise a bit and do the work once and for all. Thus, with F , G and H chosen from the set $\{\lambda, A\}$, we find by means of the chain rule and using ghost number conservation, together with the identities (64)

$$\frac{\delta}{\delta F^{bj}(y)} \frac{\delta^2 W}{\delta J_G^{ck}(x) \delta J_H^{dl}(x')} \Big|_{J=I=0} = \int dv \left[\frac{\delta^2 \Gamma}{\delta F^{bj}(y) \delta K^{em}(v)} \frac{\delta^3 W}{\delta J_K^{em}(v) \delta J_G^{ck}(x) \delta J_H^{dl}(x')} \right]_{J=I=0}$$

The field index K in this equation is summed over the two values λ and A . Expressing the connected three-point function by its one-particle irreducible counterpart

$$\begin{aligned}
W^{(FGH)abc}_{ijk}(x, y, z) = & - \int du du' du'' G^{(FF')}_{ii'}(x, u) G^{(GG')}_{jj'}(y, u') \\
& G^{(HH')}_{kk'}(z, u'') \Gamma^{(F'G'H')}_{i'j'k'}(u, u', u'') \tag{140}
\end{aligned}$$

and exploiting the relation (115) then leads to the identity

$$\begin{aligned}
& \frac{\delta}{\delta F^{bj}(y)} \frac{\delta^2 W}{\delta J_G^{ck}(x) \delta J_H^{dl}(x')} \Big|_{J=I=0} = \\
& - \int du' du'' G^{(GG')}_{kk'}(x, u') G^{(HH')}_{ll'}(x', u'') \Gamma^{(FG'H')}_{jk'l'}(y, u', u'') \tag{141}
\end{aligned}$$

Again, doubled field indices are summed over λ and A (which we will assume from now on in all relevant cases). Finally, transforming into momentum space and inserting the definition of the three-point vertex function (125) yields

$$\begin{aligned}
& \frac{\delta}{\delta F^{bj}(y)} \frac{\delta^2 W}{\delta J_G^{ck}(x) \delta J_H^{dl}(x')} \Big|_{J=I=0} = \int \frac{d^D k}{(2\pi)^D} \frac{d^D k'}{(2\pi)^D} e^{-ik(x-y)} e^{ik'(x-x')} G^{(GG')}_{kk'}(k - k') \\
& G^{(HH')}_{ll'}(k') \Gamma^{(FG'H')}_{jk'l'}(k' - k, -k') \tag{142}
\end{aligned}$$

Analogously, one can derive a general expression for the fourth functional derivative in Eq. (139). Using the chain rule as above, exploiting ghost number conservation and the identity (115), translating connected into one-particle irreducible quantities as in Eq. (140) and finally introducing the momentum space vertex functions (125) and (126) leads to

$$\begin{aligned}
& \frac{\delta}{\delta E^{bj}(y)} \frac{\delta^3 W}{\delta J_F^{ck}(x) \delta J_G^{dl}(x) \delta J_H^{em}(x)} \Big|_{J=I=0} = \int \frac{d^D k}{(2\pi)^D} e^{-ik(x-y)} \int \frac{d^D k'}{(2\pi)^D} \frac{d^D k''}{(2\pi)^D} \Big[\\
& + G^{(FF')}_{kk'}(k-k') G^{(GG')}_{ll'}(k'-k'') G^{(HH')}_{mm'}(k'') \Gamma^{(L'G'H')}_{s'l'm'}(k''-k', -k'') \\
& \quad G^{(L'K')}_{s'r'}(k') \Gamma^{(EF'K')}_{jk'r'}(k'-k, -k') \\
& + G^{(GF')}_{lk'}(k-k') G^{(HG')}_{ml'}(k'-k'') G^{(FH')}_{km'}(k'') \Gamma^{(L'G'H')}_{s'l'm'}(k''-k', -k'') \\
& \quad G^{(L'K')}_{s'r'}(k') \Gamma^{(EF'K')}_{jk'r'}(k'-k, -k') \\
& + G^{(HF')}_{mk'}(k-k') G^{(FG')}_{kl'}(k'-k'') G^{(GH')}_{lm'}(k'') \Gamma^{(L'G'H')}_{s'l'm'}(k''-k', -k'') \\
& \quad G^{(L'K')}_{s'r'}(k') \Gamma^{(EF'K')}_{jk'r'}(k'-k, -k') \\
& + G^{(FF')}_{kk'}(k-k'-k'') G^{(GG')}_{ll'}(k') \\
& \quad G^{(HH')}_{mm'}(k'') \Gamma^{(EF'G'H')}_{jk'l'm'}(k'+k''-k, -k', -k'') \Big] \quad (143)
\end{aligned}$$

Exploiting the identities (142) and (143) one can now readily obtain the DYSON–SCHWINGER equation of the $\Pi^{(\lambda\lambda)}$ self-energy component from Eq. (139). One finds

$$\begin{aligned}
\Pi^{(\lambda\lambda)}_{ij}{}^{ab}(k) = & -\frac{1}{2} \int \frac{d^D k'}{(2\pi)^D} (-g) V_{ikl}^{acd}(\mathbf{k} - \mathbf{k}', \mathbf{k}') G^{(AG')}_{kk'}(k-k') G^{(AH')}_{ll'}(k') \\
& \quad \Gamma^{(\lambda G'H')}_{jk'l'}(k'-k, -k') \\
& -\frac{1}{2} \int \frac{d^D k'}{(2\pi)^D} \frac{d^D k''}{(2\pi)^D} ig^2 V_{iklm}^{acde} G^{(AF')}_{kk'}(k-k') G^{(AG')}_{ll'}(k'-k'') G^{(AH')}_{mm'}(k'') \\
& \quad \Gamma^{(L'G'H')}_{s'l'm'}(k''-k', -k'') G^{(L'K')}_{s'r'}(k') \Gamma^{(\lambda F'K')}_{jk'r'}(k'-k, -k') \\
& -\frac{1}{6} \int \frac{d^D k'}{(2\pi)^D} \frac{d^D k''}{(2\pi)^D} ig^2 V_{iklm}^{acde} G^{(AF')}_{kk'}(k-k'-k'') G^{(AG')}_{ll'}(k') G^{(AH')}_{mm'}(k'') \\
& \quad \Gamma^{(\lambda F'G'H')}_{jk'l'm'}(k'+k''-k, -k', -k'') \quad (144)
\end{aligned}$$

where we have used the symmetry of the vertex V_{iklm}^{acde} in the last three pairs of indices to combine the first three terms arising from Eq. (143) into one. We have illustrated Eq. (144) in Fig. 2.

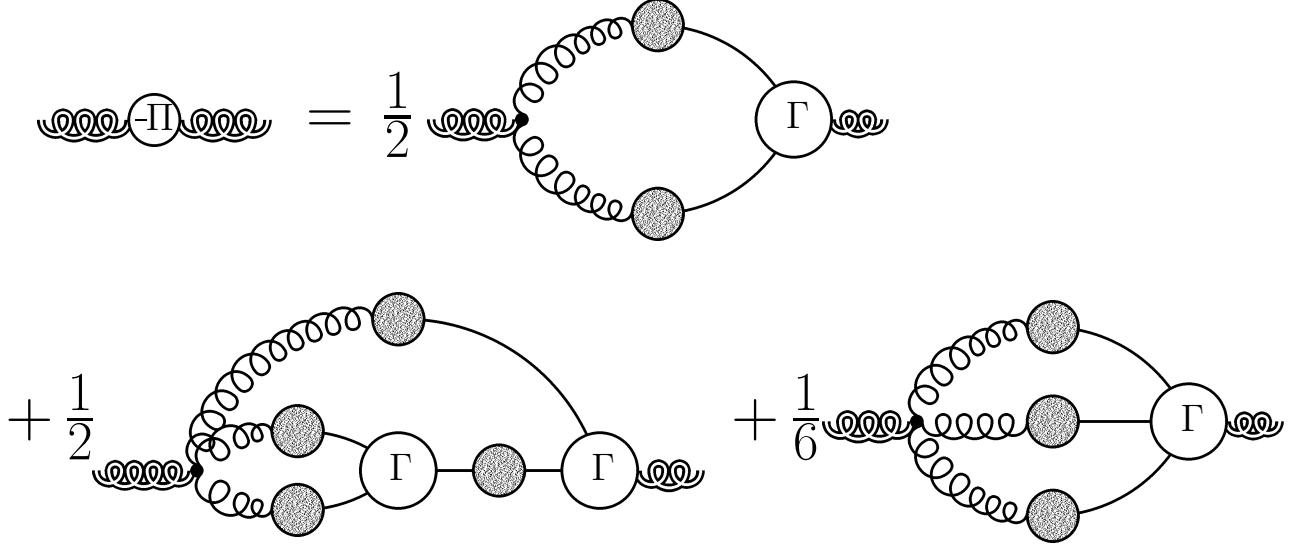


FIG. 2: DYSON-SCHWINGER equation of the $\Pi^{(\lambda\lambda)}$ self-energy component, Eq. (144).

4. DSE for $\Pi^{(\lambda A)}(k)$

Taking the derivative of Eq. (100) with respect to $A^{bj}(y)$ instead of $\lambda^{bj}(y)$ and afterwards setting the sources to zero leads to the DYSON-SCHWINGER equation for the $\Pi^{(\lambda A)}$ self-energy component, namely

$$\begin{aligned}
 \frac{\delta^2 \Gamma}{\delta \lambda^{ai}(x) \delta A^{bj}(y)} \Big|_{J=I=0} &= \overbrace{-i \delta^{ab} \left[(+\sigma \partial_t - \Delta) \delta_{ij} + (1 - \frac{\sigma}{\kappa}) \partial_i \partial_j \right] \delta(x - y)}^{\substack{(\Delta^{-1})^{(\lambda A)}{}_{ij}^{ab}(x, y)}} \\
 &\quad - \frac{ig}{2!} f^{acd} \left[(1 - \frac{\sigma}{\kappa}) [\delta^{ik} \partial'_l - \delta^{il} \partial'_k] + 2 [\delta^{ik} \partial_l - \delta^{il} \partial'_k] \right. \\
 &\quad \left. + [\delta^{kl} \partial'_i - \delta^{lk} \partial'_i] \right] \frac{\delta}{\delta A^{bj}(y)} \frac{\delta^2 W}{\delta J_A^{ck}(x) \delta J_A^{dl}(x')} \Big|_{x' = x} \\
 &\quad J = I = 0 \\
 &\quad - \frac{ig^2}{3!} V_{iklm}^{acde} \frac{\delta}{\delta A^{bj}(y)} \frac{\delta^3 W}{\delta J_A^{ck}(x) \delta J_A^{dl}(x) \delta J_A^{em}(x)} \Big|_{J=I=0} \\
 &\quad - \frac{ig^2}{2!} V_{iklj}^{acdb} \delta(x - y) \frac{\delta^2 W}{\delta J_A^{ck}(x) \delta J_A^{dl}(x)} \Big|_{J=I=0} \tag{145}
 \end{aligned}$$

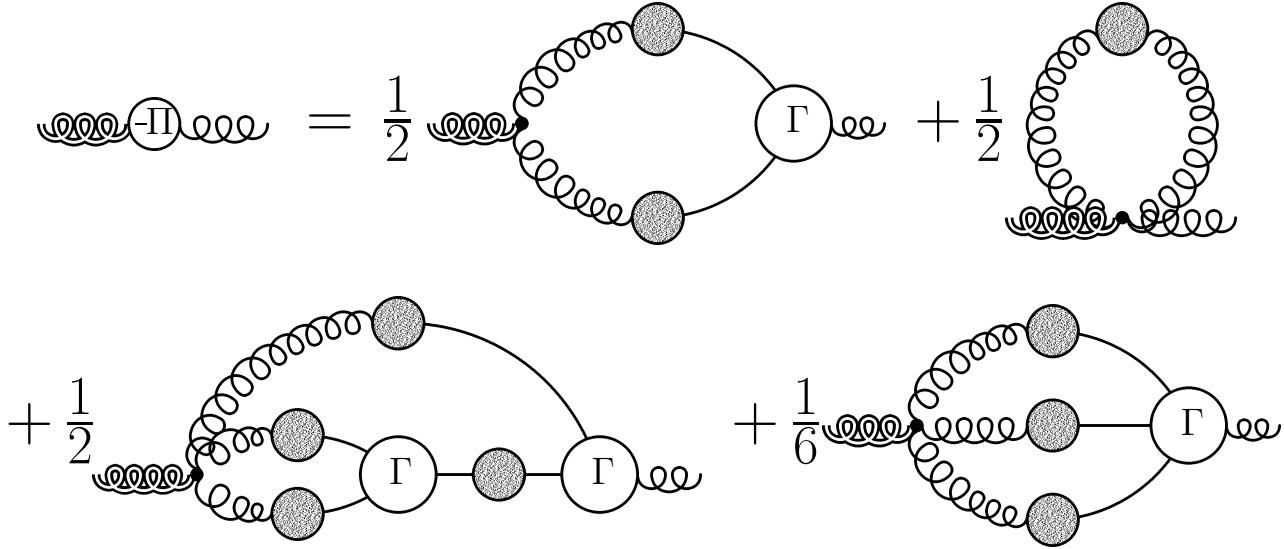


FIG. 3: DYSON–SCHWINGER equation of the $\Pi^{(\lambda A)}$ self-energy component, Eq. (146).

Reading off the self-energy component by comparing with Eq. (118), and using Eqs. (142)–(143) one arrives at

$$\begin{aligned}
 \Pi_{ij}^{(\lambda A)ab}(k) = & -\frac{1}{2} \int \frac{d^D k'}{(2\pi)^D} (-g) V_{ikl}^{acd}(\mathbf{k} - \mathbf{k}', \mathbf{k}') G^{(AG')cc'}_{kk'}(k - k') G^{(AH')dd'}_{ll'}(k') \\
 & \Gamma^{(AG' H')}_{jk'l'}(k' - k, -k') \\
 & -\frac{1}{2} \int \frac{d^D k'}{(2\pi)^D} \frac{d^D k''}{(2\pi)^D} ig^2 V_{iklm}^{acde} G^{(AF')cc'}_{kk'}(k - k') G^{(AG')dd'}_{ll'}(k' - k'') G^{(AH')ee'}_{mm'}(k'') \\
 & \Gamma^{(L'G' H')}_{s'l'm'}(k'' - k', -k'') G^{(L'K')}_{s'r'}(k') \Gamma^{(AF' K')}_{jk'r'}(k' - k, -k') \\
 & -\frac{1}{6} \int \frac{d^D k'}{(2\pi)^D} \frac{d^D k''}{(2\pi)^D} ig^2 V_{iklm}^{acde} G^{(AF')cc'}_{kk'}(k - k' - k'') G^{(AG')dd'}_{ll'}(k') G^{(AH')ee'}_{mm'}(k'') \\
 & \Gamma^{(AF' G' H')}_{jk'l'm'}(k' + k'' - k, -k', -k'') \\
 & -\frac{1}{2} \int \frac{d^D k'}{(2\pi)^D} ig^2 V_{ijkl}^{abcd} G^{(AA)cd}_{kl}(k') \tag{146}
 \end{aligned}$$

which is depicted in Fig. 3.

5. DSE for $\Pi^{(A\lambda)}(k)$

From the gauge field equation (106) one obtains by taking the derivative with respect to $\lambda^{bj}(y)$ for vanishing sources

$$\begin{aligned}
& \frac{\delta^2 \Gamma}{\delta A^{ai}(x) \delta \lambda^{bj}(y)} \Big|_{J=I=0} = \overbrace{-i \delta^{ab} [(-\sigma \partial_t - \Delta) \delta_{ij} + (1 - \frac{\sigma}{\kappa}) \partial_i \partial_j] \delta(x-y)}^{\overbrace{(\Delta^{-1})^{(A\lambda)ab}_{ij}(x,y)}} \\
& \quad - ig f^{acd} \left[-\left(1 - \frac{\sigma}{\kappa}\right) [\delta^{ik} \partial_l' + \delta^{kl} (\partial_i + \partial_i')] + 2 [\delta^{kl} \partial_i' + \delta^{ik} (\partial_l + \partial_l')] \right. \\
& \quad \left. - [\delta^{il} \partial_k' + \delta^{il} (\partial_k + \partial_k')] \right] \frac{\delta}{\delta \lambda^{bj}(y)} \frac{\delta^2 W}{\delta J_{\lambda}^{ck}(x) \delta J_A^{dl}(x')} \Big|_{x'=x} \\
& \quad - \frac{ig^2}{2!} V_{mikl}^{eacd} \frac{\delta}{\delta \lambda^{bj}(y)} \frac{\delta^3 W}{\delta J_{\lambda}^{em}(x) \delta J_A^{ck}(x) \delta J_A^{dl}(x)} \Big|_{J=I=0} \\
& \quad - \frac{ig^2}{2!} V_{jikl}^{bacd} \delta(x-y) \frac{\delta^2 W}{\delta J_A^{ck}(x) \delta J_A^{dl}(x)} \Big|_{J=I=0} \\
& \quad - \frac{g}{\kappa} f^{acd} \partial_i \frac{\delta}{\delta \lambda^{bj}(y)} \frac{\delta^2 W}{\delta J_{\omega}^c(x') \delta J_{\omega}^d(x)} \Big|_{x'=x} \\
& \quad \Big|_{J=I=0} \tag{147}
\end{aligned}$$

As in Eq. (146, we have again a self-energy, a tadpole, and terms of the type in Eqs. (142)–(143), but we also have a new term involving gauge ghosts. It can be calculated in a similar way to the previous cases, and comes out to

$$\begin{aligned}
& \frac{\delta}{\delta E^{bj}(y)} \frac{\delta^2 W}{\delta J_{\omega}^c(x') \delta J_{\omega}^d(x)} \Big|_{J=I=0} = - \int \frac{d^D k}{(2\pi)^D} \frac{d^D k'}{(2\pi)^D} e^{-ik(x-y)} e^{ik'(x-x')} G^{(\omega)c'c}(-k') \\
& \quad G^{(\omega)dd'}(k-k') \Gamma^{(\bar{\omega}\omega E)d'c'b}_{j}(k'-k, -k') \tag{148}
\end{aligned}$$

With this, and the previous identities, Eq. (147) can be written as

$$\begin{aligned}
& \Pi^{(A\lambda)ab}_{ij}(k) = - \int \frac{d^D k'}{(2\pi)^D} (-g) V_{kli}^{cda}(\mathbf{k}', -\mathbf{k}) G^{(\lambda A)cc'}_{kk'}(k-k') G^{(AH')dd'}_{ll'}(k') \\
& \quad \Gamma^{(\lambda AH')bc'd'}_{jk'l'}(k'-k, -k') \\
& \quad - \int \frac{d^D k'}{(2\pi)^D} \frac{d^D k''}{(2\pi)^D} ig^2 V_{mikl}^{eacd} G^{(AF')cc'}_{kk'}(k-k') G^{(AG')dd'}_{ll'}(k'-k'') G^{(\lambda A)ee'}_{mm'}(k'') \\
& \quad \Gamma^{(L'G'A)h'd'e'}_{s'l'm'}(k''-k', -k'') G^{(L'K')h'g'}_{s'r'}(k') \Gamma^{(\lambda F'K')bc'g'}_{jk'r'}(k'-k, -k') \\
& \quad - \frac{1}{2} \int \frac{d^D k'}{(2\pi)^D} \frac{d^D k''}{(2\pi)^D} ig^2 V_{mikl}^{eacd} G^{(\lambda A)ec'}_{mk'}(k-k') G^{(AG')cd'}_{kl'}(k'-k'') G^{(AH')de'}_{lm'}(k'') \\
& \quad \Gamma^{(L'G'H')h'd'e'}_{s'l'm'}(k''-k', -k'') G^{(L'K')h'g'}_{s'r'}(k') \Gamma^{(\lambda AK')bc'g'}_{jk'r'}(k'-k, -k') \\
& \quad - \frac{1}{2} \int \frac{d^D k'}{(2\pi)^D} \frac{d^D k''}{(2\pi)^D} ig^2 V_{mikl}^{eacd} G^{(AF')cc'}_{kk'}(k-k'-k'') G^{(AG')dd'}_{ll'}(k') G^{(\lambda A)ee'}_{mm'}(k'') \\
& \quad \Gamma^{(\lambda F'G'A)bc'd'e'}_{jk'l'm'}(k'+k''-k, -k', -k'') \\
& \quad + \int \frac{d^D k'}{(2\pi)^D} \frac{ig}{\kappa} f^{cda}(k-k')^i G^{(\omega)c'c}(-k') G^{(\omega)dd'}(k-k') \Gamma^{(\bar{\omega}\omega\lambda)d'c'b}_{j}(k'-k, -k') \\
& \quad - \frac{1}{2} \int \frac{d^D k'}{(2\pi)^D} ig^2 V_{jikl}^{bacd} G^{(AA)cd}_{kl}(k') \tag{149}
\end{aligned}$$

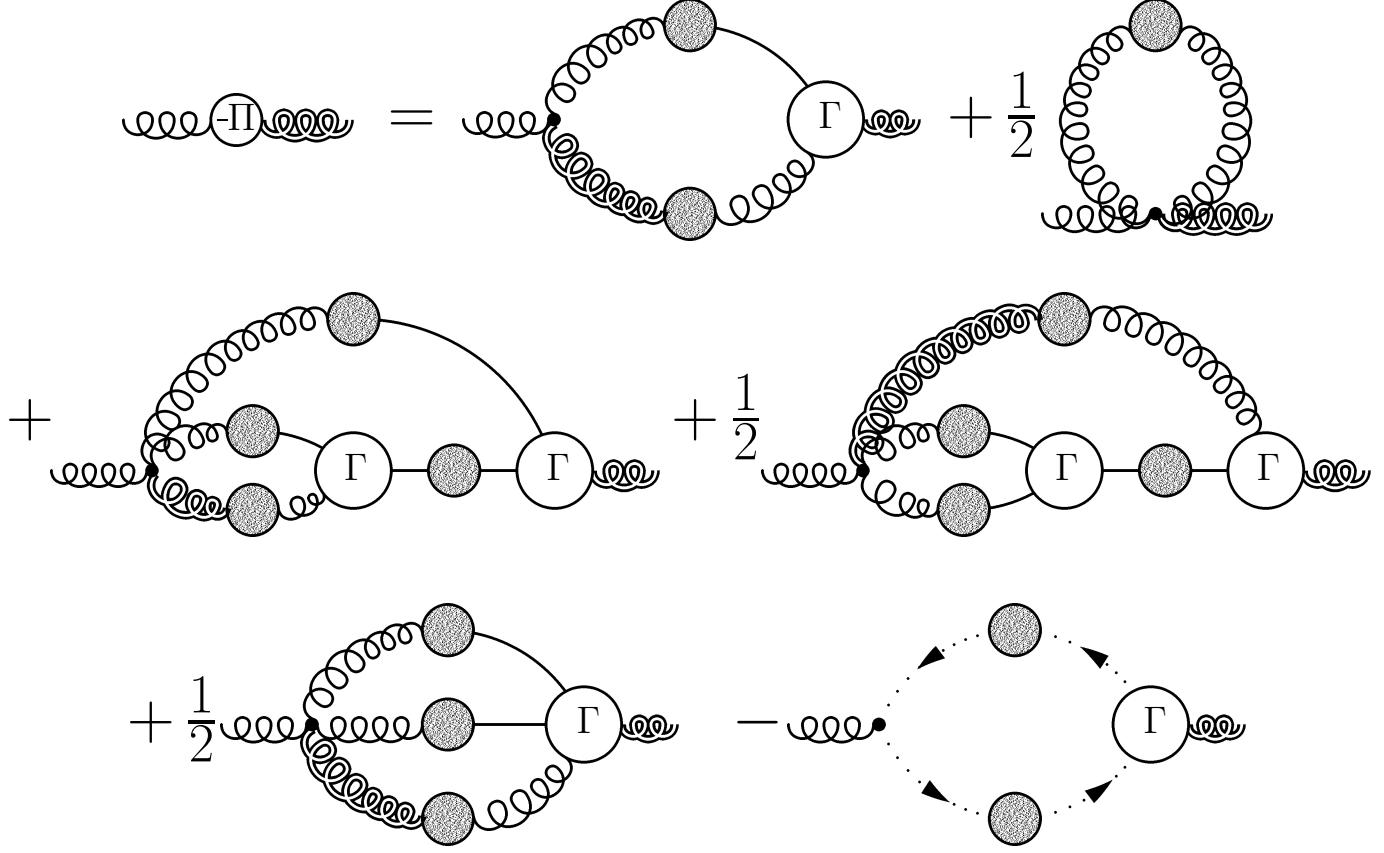


FIG. 4: DYSON-SCHWINGER equation of the $\Pi^{(AA)}$ self-energy component, Eq. (149).

A graphical representation of this identity can be found in Fig. 4.

6. DSE for $\Pi^{(AA)}(k)$

Finally, we come to the pure gauge field component $\Pi^{(AA)}$. Because $(\Delta^{-1})^{(AA)ab}_{ij} = 0$, one has in this case

$$\Pi_{ij}^{(AA)ab}(x, y) = \frac{\delta^2 \Gamma}{\delta A^{ai}(x) \delta A^{bj}(y)} \Big|_{J=I=0} \quad (150)$$

and thus one obtains from Eq. (106) the final identity

$$\begin{aligned}
\Pi^{(AA)}_{ij}^{ab}(k) = & - \int \frac{d^D k'}{(2\pi)^D} (-g) V_{kli}^{cda}(\mathbf{k}', -\mathbf{k}) G^{(\lambda A)}_{kk'}^{cc'}(k - k') G^{(A\lambda)}_{ll'}^{dd'}(k') \\
& \Gamma^{(AA\lambda)}_{jk'l'}^{bc'd'}(k' - k, -k') \\
& - \int \frac{d^D k'}{(2\pi)^D} \frac{d^D k''}{(2\pi)^D} ig^2 V_{mikl}^{eacd} G^{(AF')}_{kk'}^{cc'}(k - k') G^{(AG')}_{ll'}^{dd'}(k' - k'') G^{(\lambda A)}_{mm'}^{ee'}(k'') \\
& \Gamma^{(L'G'A)}_{s'l'm'}^{h'd'e'}(k'' - k', -k'') G^{(L'K')}_{s'r'}^{h'g'}(k') \Gamma^{(AF'K')}_{jk'r'}^{bc'g'}(k' - k, -k') \\
& - \frac{1}{2} \int \frac{d^D k'}{(2\pi)^D} \frac{d^D k''}{(2\pi)^D} ig^2 V_{mikl}^{eacd} G^{(\lambda A)}_{mk'}^{ec'}(k - k') G^{(AG')}_{kl'}^{cd'}(k' - k'') G^{(AH')}_{lm'}^{de'}(k'') \\
& \Gamma^{(AG'H')}_{s'l'm'}^{h'd'e'}(k'' - k', -k'') G^{(A\lambda)}_{s'r'}^{h'g'}(k') \Gamma^{(AA\lambda)}_{jk'r'}^{bc'g'}(k' - k, -k') \\
& - \frac{1}{2} \int \frac{d^D k'}{(2\pi)^D} \frac{d^D k''}{(2\pi)^D} ig^2 V_{mikl}^{eacd} G^{(AF')}_{kk'}^{cc'}(k - k' - k'') G^{(AG')}_{ll'}^{dd'}(k') G^{(\lambda A)}_{mm'}^{ee'}(k'') \\
& \Gamma^{(AF'G'A)}_{jk'l'm'}^{bc'd'e'}(k' + k'' - k, -k', -k'') \\
& + \int \frac{d^D k'}{(2\pi)^D} \frac{ig}{\kappa} f^{cda}(k - k')^i G^{(\omega)}_{c'c}(-k') G^{(\omega)}_{dd'}(k - k') \Gamma^{(\bar{\omega}\omega A)}_{j}^{d'c'b}(k' - k, -k') \\
& - \int \frac{d^D k'}{(2\pi)^D} ig^2 V_{mikj}^{eacb} G^{(\lambda A)}_{mk}^{ec}(k') \tag{151}
\end{aligned}$$

which completes our derivation of the DYSON–SCHWINGER equations in Bödeker’s effective theory.

V. DISCUSSION AND OUTLOOK

In this work we have constructed an analytic approach to the non-perturbative physics encoded in Bödeker’s effective theory [1, 8]. Our approach is based on DYSON–SCHWINGER equations and allows for an investigation of the non-perturbative dynamics of soft, non-abelian hot gauge fields that is independent of the existing lattice studies of Bödeker’s theory [18, 19].

The basic starting point is to transform Bödeker’s Langevin equation into a path integral. From this path integral, in principle, one could deduce the DYSON–SCHWINGER equations. However, it would hardly be avoidable to introduce an uncontrolled gauge dependence when finally truncating these equations. To control this gauge dependence, we therefore enlarged the system by the introduction of gauge ghosts (which is optional in stochastic quantisation). This enlarged system is endowed with a BRST symmetry reflecting the gauge invariance; and we have derived the corresponding WARD–TAKAHASHI

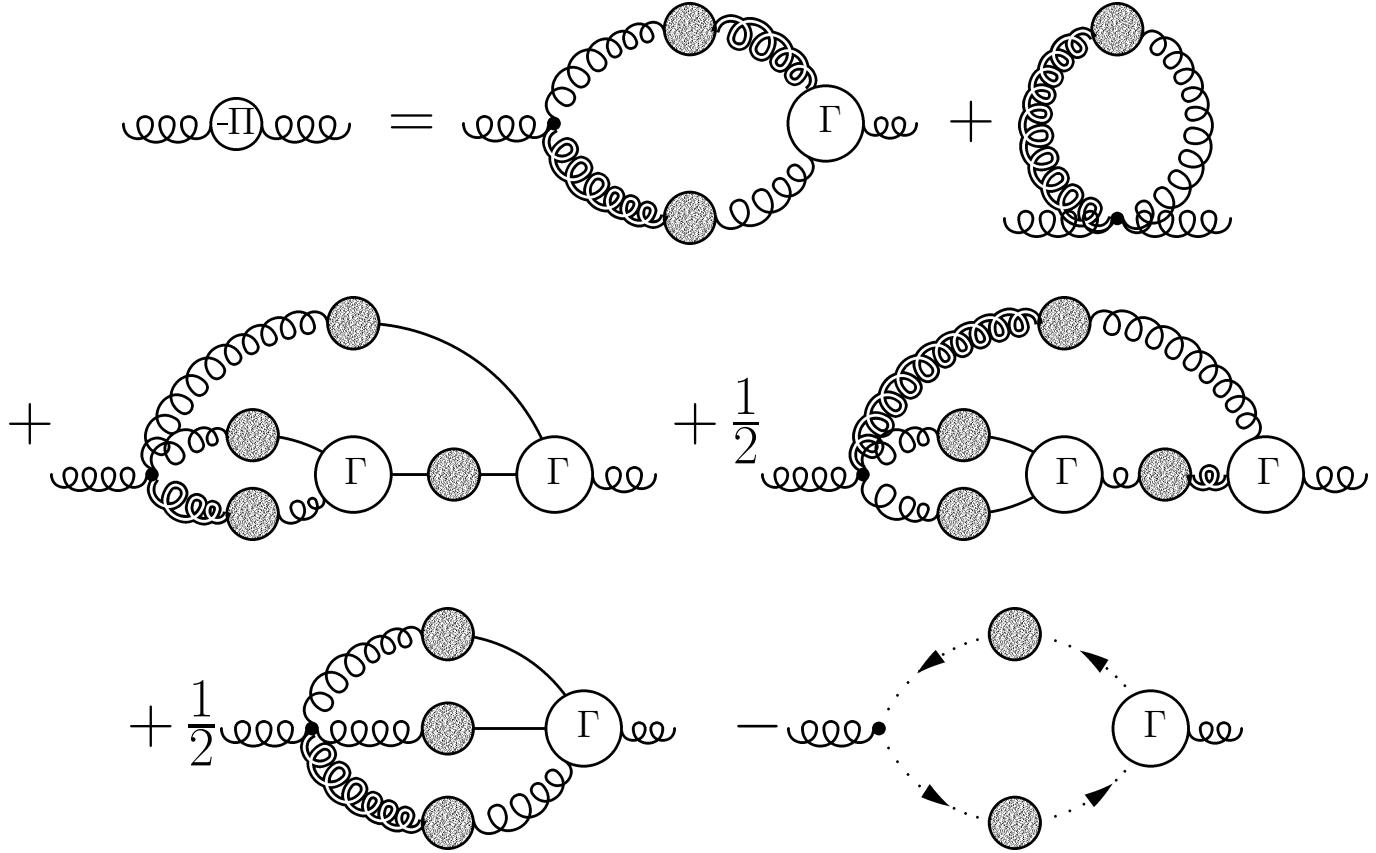


FIG. 5: DYSON–SCHWINGER equation of the $\Pi^{(AA)}$ self-energy component, Eq. (151).

identities. A consistent truncation of the DYSON–SCHWINGER equations is achieved if the gauge and ghost sectors are truncated in accordance with these identities.

We also derived a second class of restrictions, so-called stochastic Ward identities known from stochastic quantisation [12]. These reflect the characteristic structure of the path integral action induced by its origin in a stochastic differential equation.

Finally, we have deduced the DYSON–SCHWINGER equations of the theory. They contain, in principle, the possibility of (finite!) vertices coupling auxiliary fields to gauge ghosts or gauge field/auxiliary field vertices with more than one auxiliary field, both of which are not present at tree level. Whether these vertices are really non-zero, will be an interesting question to be decided by an implementation of our formalism.

In combination with the gauge and stochastic Ward identities given in Eqs. (C.25) – (C.27), the DYSON–SCHWINGER equations (136), (137), (144), (146), (149) and (151) provide all the necessary tools for an analytic study of the non-perturbative physics encoded in Bödeker’s effective theory. In particular, it can be used to study the sphaleron rate Eq. (1),

where N_{CS} in terms of the gauge field takes the form

$$N_{\text{CS}}(t_2) - N_{\text{CS}}(t_1) = \int_{t_1}^{t_2} dt \int d^3x \frac{g^2}{8\pi^2} E_i^a B_i^a(x) \quad (152)$$

Restricting to the lowest correlators, we are then interested in the unequal time correlators $\langle E_i^a(x_1) E_j^b(x_2) \rangle$, $\langle E_i^a(x_1) B_j^b(x_2) \rangle$, and $\langle B_i^a(x_1) B_j^b(x_2) \rangle$. The first one should approach a delta function, while the second one should be subleading [20]. It would be a good test of our ansatz if we could (roughly) reproduce the factor in front of the sphaleron rate [18, 19].

We close this discussion with a few comments on what is to come. Since for the hot sphaleron rate we are interested primarily in the infrared behaviour of the theory, the first thing to be done moving forward is determining the appropriate relation between the anomalous dimensions for k_0 and $|\mathbf{k}|^2$. This can be done by investigating the limit when $k_0 \rightarrow 0$, and comparing with the anomalous dimension in YANG-MILLS theory in three dimensions [21]. One can also analyse the importance of the ansatz for the vertex functions by comparing with time-independent stochastic quantisation [13].

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APPENDIX A: CALCULATION OF JACOBIANS

Throughout this work, there appear several times Jacobians as products of change of variables. As is well known from the literature [15], we have claimed that they are constants and have generally absorbed them in the measure. To make this work more self-contained, we provide here a derivation of this claim.

In order to simplify the expressions, we will suppress the colour and space indices until it becomes necessary. The first Jacobian that we encountered was in Eq. (5), where the following expression appears

$$\text{Det} \left(\frac{\delta \mathbf{E}}{\delta \mathbf{A}} \right) \quad (\text{A.1})$$

with

$$\frac{\delta \mathbf{E}[\mathbf{A}]}{\delta \mathbf{A}} = \frac{\partial}{\partial t} + \frac{1}{2} \frac{\delta \mathbf{K}[\mathbf{A}]}{\delta \mathbf{A}} = \frac{\partial}{\partial t} \left(\mathbb{1} + \frac{1}{2} \left(\frac{\partial}{\partial t} \right)^{-1} \frac{\delta \mathbf{K}[\mathbf{A}]}{\delta \mathbf{A}} \right) \quad (\text{A.2})$$

where \mathbf{K} contains all the terms in the left-hand side of Eq.(12) without time derivatives. The kernel of the operator $(\partial/\partial t)^{-1}$ is constrained by causality to be $\Theta(t_2 - t_1)$. We then have

$$\text{Det} \left(\frac{\delta \mathbf{E}}{\delta \mathbf{A}} \right) = \text{const.} \cdot \text{Det} \left(\mathbb{1} + \frac{1}{2} \left(\frac{\partial}{\partial t} \right)^{-1} \frac{\delta \mathbf{K}[\mathbf{A}]}{\delta \mathbf{A}} \right) \quad (\text{A.3})$$

with

$$\left[\frac{1}{2} \left(\frac{\partial}{\partial t} \right)^{-1} \frac{\delta \mathbf{K}[\mathbf{A}]}{\delta \mathbf{A}} \right]_{ij}^{ab}(t, \mathbf{x}; t', \mathbf{x}') = \frac{1}{2} \int dt'' \Theta(t - t'') \frac{\delta K_i^a[\mathbf{A}](t'', \mathbf{x})}{\delta A_j^b(t', \mathbf{x}')} \quad (\text{A.4})$$

Since \mathbf{K} contains no time derivatives, the functional derivative produces a delta function in the time variable i.e.

$$\frac{\delta K_i^a[\mathbf{A}](t'', \mathbf{x})}{\delta A_j^b(t', \mathbf{x}')} = \delta(t'' - t') \frac{\delta_x K_i^a[\mathbf{A}](t', \mathbf{x})}{\delta_x A_j^b(t', \mathbf{x}')} \quad (\text{A.5})$$

where we have introduced the symbol δ_x to denote a variation with respect to the \mathbf{x} dependence only. Hence, we find

$$\left[\frac{1}{2} \left(\frac{\partial}{\partial t} \right)^{-1} \frac{\delta K[\phi]}{\delta \phi} \right]_{\alpha\beta}(t, \mathbf{x}; t', \mathbf{x}') = \frac{1}{2} \Theta(t - t') \frac{\delta_x K_i^a[\phi](t', \mathbf{x})}{\delta_x A_j^b(t', \mathbf{x}')} \quad (\text{A.6})$$

Coming back to Eq. (A.3) and using $\text{Tr} \ln(\dots) = \ln \text{Det}(\dots)$ in addition to the series expansion of the logarithm, the determinant takes the form

$$\text{Det} \left(\frac{\delta \mathbf{E}[\mathbf{A}]}{\delta \mathbf{A}} \right) = \text{const.} \cdot \exp \left\{ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \frac{1}{2^n} \text{Tr} \left[\left(\frac{\partial}{\partial t} \right)^{-1} \frac{\delta \mathbf{K}[\mathbf{A}]}{\delta \mathbf{A}} \right]^n \right\} \quad (\text{A.7})$$

The trace in this expression can be evaluated with the help of Eq. (A.6). One obtains

$$\begin{aligned} \text{Tr} \left[\left(\frac{\partial}{\partial t} \right)^{-1} \frac{\delta \mathbf{K}[\mathbf{A}]}{\delta \mathbf{A}} \right]^n &= \int dt_1 \cdots dt_n d^{D-1}x_1 \cdots d^{D-1}x_n \Theta(t_1 - t_2) \frac{\delta_x K_{i_1}^{a_1}[\mathbf{A}](t_2, \mathbf{x}_1)}{\delta_x A_{i_2}^{a_2}(t_2, \mathbf{x}_2)} \\ &\quad \Theta(t_2 - t_3) \frac{\delta_x K_{i_2}^{a_2}[\mathbf{A}](t_3, \mathbf{x}_2)}{\delta_x A_{i_3}^{a_3}(t_3, \mathbf{x}_3)} \cdots \Theta(t_n - t_1) \frac{\delta_x K_{i_n}^{a_n}[\mathbf{A}](t_1, \mathbf{x}_n)}{\delta_x A_{i_1}^{a_1}(t_1, \mathbf{x}_1)} \end{aligned} \quad (\text{A.8})$$

and thus

$$\text{Tr} \left[\left(\frac{\partial}{\partial t} \right)^{-1} \frac{\delta K[\phi]}{\delta \phi} \right]^n = \int dt_1 \cdots dt_n \Theta(t_1 - t_2) \Theta(t_2 - t_3) \cdots \Theta(t_n - t_1) f_n(t_1, t_2, \dots, t_n) \quad (\text{A.9})$$

if we set

$$f_n(t_1, t_2, \dots, t_n) = \int d^{D-1}x_1 \cdots d^{D-1}x_n \frac{\delta_x K_{i_1}^{a_1}[\mathbf{A}](t_2, \mathbf{x}_1)}{\delta_x A_{i_2}^{a_2}(t_2, \mathbf{x}_2)} \frac{\delta_x K_{i_2}^{a_2}[\mathbf{A}](t_3, \mathbf{x}_2)}{\delta_x A_{i_3}^{a_3}(t_3, \mathbf{x}_3)} \cdots \frac{\delta_x K_{i_n}^{a_n}[\mathbf{A}](t_1, \mathbf{x}_n)}{\delta_x A_{i_1}^{a_1}(t_1, \mathbf{x}_1)}$$

for abbreviation. Unless $n = 1$, however, the expression (A.9) vanishes for any function f_n . Therefore, only the first term of the sum in Eq. (A.7) survives and we finally arrive at

$$\text{Det}\left(\frac{\delta \mathbf{E}[\mathbf{A}]}{\delta \mathbf{A}}\right) = \text{const.} \cdot \exp\left\{\frac{1}{2} \Theta(0) \int dt d^{D-1}x \frac{\delta_x K_i^a[\mathbf{A}](t, \mathbf{x})}{\delta_x A_b^a(t, \mathbf{x}')}\Big|_{\mathbf{x}'=\mathbf{x}}\right\} \quad (\text{A.10})$$

Our next task is to calculate the functional derivative of $K_i^a[\mathbf{A}]$. To this end, it is easiest to write it down in components which clarifies the structure

$$\begin{aligned} \frac{1}{2} K_i^a[\mathbf{A}] &= g f^{abc} \left[\left(1 - \frac{\sigma}{\kappa}\right) A_i^b \partial_j A_j^c + 2 A_j^c \partial_j A_i^b + A_j^b \partial_i A_j^c \right] \\ &+ \left[\left(1 - \frac{\sigma}{\kappa}\right) \partial_i \partial_j - \delta_{ij} \Delta \right] A_j^a + g^2 f^{abc} f^{bde} A_j^c A_j^d A_i^e \end{aligned} \quad (\text{A.11})$$

Obviously, the first term, i.e. the term quadratic in the gauge field, does not contribute to the functional derivative with respect to A_i^a because it always produces a δ^{ab} or δ^{ac} that is contracted with the structure constants f^{abc} in front of the square bracket. The linear term, on the other hand, only contributes a constant that can be absorbed into the constant in Eq. (A.10). Thus, we only have to take care of the third order term which leads to

$$\text{Det}\left(\frac{\delta \mathbf{E}[\mathbf{A}]}{\delta \mathbf{A}}\right) = \text{const.}' \cdot \exp\left\{ C_A (D-2) \Theta(0) \delta^{D-1}(0) \frac{g^2}{\sigma} \int dt d^{D-1}x \mathbf{A}^a(t, \mathbf{x}) \cdot \mathbf{A}^a(t, \mathbf{x}) \right\} \quad (\text{A.12})$$

where $f^{acd} f^{bcd} = C_A \delta^{ab}$ as usual. However, in dimensional regularisation $\delta^{D-1}(0)$ gives zero as a consequence of the general rules of D-dimensional integration, and the determinant is simply a constant.

We came across another determinant in Eq. (34)

$$\text{Det}\left(\frac{\delta^\omega \zeta}{\delta \zeta}\right) \quad (\text{A.13})$$

One finds

$$\frac{\delta^\omega \zeta^{ai}(t, \mathbf{x})}{\delta \zeta^{bj}(t', \mathbf{x}')} = \delta^{ab} \delta_{ij} \delta(t - t') \delta^{D-1}(\mathbf{x} - \mathbf{x}') + g f^{acd} \frac{\delta}{\delta \zeta^{bj}(t', \mathbf{x}')} [\omega^d[\zeta](t, \mathbf{x}) \zeta^{ci}(t, \mathbf{x})] \quad (\text{A.14})$$

and thus because ω is infinitesimal

$$\text{Det}\left(\frac{\delta^\omega \zeta}{\delta \zeta}\right) = 1 + \int dt d^{D-1}x g f^{acd} \left[\frac{\delta}{\delta \zeta^{ai}(t', \mathbf{x}')} [\omega^d[\zeta](t, \mathbf{x}) \zeta^{ci}(t, \mathbf{x})] \right]_{\substack{t' = t \\ \mathbf{x}' = \mathbf{x}}} \quad (\text{A.15})$$

The functional derivative acting on ζ^{ci} produces a δ^{ac} and therefore does not contribute because the Kronecker delta is contracted with the structure constants. To determine the

remaining functional derivative of $\omega^d[\zeta]$, let us formally integrate Eq. (27)

$$\omega^a(t, \mathbf{x}) = \omega^a(-\infty, \mathbf{x}) - \int_{-\infty}^t dt'' [H[\mathbf{A}]\omega]^a(t'', \mathbf{x}) - \int_{-\infty}^t dt'' \delta v^a[\mathbf{A}](t'', \mathbf{x}) \quad (\text{A.16})$$

Since $H[\mathbf{A}]$ and $\delta v^a[\mathbf{A}]$ are local functionals in time, this equation for ω has a causal character, i.e. $\omega(t, \mathbf{x})$ does only depend on the values of the gauge field $\mathbf{A}(t'', \mathbf{x})$ at times $t'' < t$. On the other hand, Eq. (26) leads to

$$\sigma \mathbf{A}^a(t, \mathbf{x}) = \sigma \mathbf{A}^a(-\infty, \mathbf{x}) - \int_{-\infty}^t dt'' [\mathbf{D}^{ab} \times \mathbf{B}^b + \sigma \mathbf{D}^{ab} v^b[\mathbf{A}]](t'', \mathbf{x}) + \int_{-\infty}^t dt'' \zeta^a(t'', \mathbf{x}) \quad (\text{A.17})$$

and $\mathbf{A}(t, \mathbf{x})$ itself only depends on the stochastic force $\zeta(t'', \mathbf{x})$ for $t'' < t$. Hence, neither $\mathbf{A}(t, \mathbf{x})$ nor $\omega(t, \mathbf{x})$ have a dependence on $\zeta(t'', \mathbf{x})$ unless $t'' < t$ and in taking the functional derivative of Eq. (A.16), we can restrict the integration range accordingly

$$\frac{\delta \omega^a[\zeta](t, \mathbf{x})}{\delta \zeta^{bi}(t', \mathbf{x}')} = - \int_{t'}^t dt'' \frac{\delta [H[\mathbf{A}]\omega]^a(t'', \mathbf{x})}{\delta \zeta^{bi}(t', \mathbf{x}')} - \int_{t'}^t dt'' \frac{\delta \delta v^a[\mathbf{A}](t'', \mathbf{x})}{\delta \zeta^{bi}(t', \mathbf{x}')} \quad (\text{A.18})$$

Evaluating this relation for $t = t'$ as in Eq. (A.15) leads to

$$\left. \frac{\delta \omega^a[\zeta](t, \mathbf{x})}{\delta \zeta^{bi}(t', \mathbf{x}')} \right|_{t=t'} = 0 \quad (\text{A.19})$$

The only way to escape this conclusion would be an integrand that is singular in time. However, if $\delta \omega / \delta \zeta$ appearing under the integral in Eq. (A.18) was singular, the integrated expression would be finite which again is $\delta \omega / \delta \zeta$. Therefore, $\delta \omega / \delta \zeta$ can not be singular. $\delta \mathbf{A} / \delta \zeta$ on the other hand can not be singular neither because of the same argument applied to the functional derivative of Eq. (A.17) with respect to ζ . Thus, we conclude

$$\text{Det} \left(\frac{\delta \omega[\zeta]}{\delta \zeta} \right) = 1 \quad (\text{A.20})$$

which completes the proof.

During the introduction of gauge ghosts to the path integral, Eq. (42), there appears in our work another Jacobian. We can see that it has the same form as the one we have already calculated, but with

$$\frac{1}{2} K^a[\omega, \mathbf{A}](t, \mathbf{x}) = -\frac{1}{\kappa} (\mathbf{D}^{ab} \cdot \nabla \omega^b)(t, \mathbf{x}) \quad (\text{A.21})$$

Hence, we can rely on our general result for the determinant, Eq. (A.10),

$$\text{Det} \left(\frac{\delta \gamma[\omega, \mathbf{A}]}{\delta \omega} \right) = \text{const.} \cdot \exp \left\{ -\frac{1}{\kappa} \Theta(0) \int dt d^{D-1}x \left. \frac{\delta_x (\mathbf{D}^{ab} \cdot \nabla \omega^b)(t, \mathbf{x})}{\delta_x \omega^a(t, \mathbf{x}')} \right|_{\mathbf{x}'=\mathbf{x}} \right\} \quad (\text{A.22})$$

The functional derivative with respect to spacial variations is given by

$$\frac{\delta_x(\mathbf{D}^{ab} \cdot \nabla \omega^b)(t, \mathbf{x})}{\delta_x \omega^d(t, \mathbf{x}')} = (\delta^{ab} \nabla - g f^{abc} \mathbf{A}^c) \cdot \nabla \delta^D(\mathbf{x} - \mathbf{x}') \delta^{bd} \quad (\text{A.23})$$

and thus, evaluated for $d = a$, gives a constant because the \mathbf{A} dependent contribution is set to zero due to the antisymmetry of the structure constants. Note that, this time, we did not have to rely on dimensional regularisation to proof the constancy of the determinant as we had to in the case of $\text{Det}(\delta \mathbf{E}[\mathbf{A}] / \delta \mathbf{A})$.

When we performed the BRST transformation in our derivation of the Ward identities, Eq. (49), one more type of determinant appeared. In general, if x_a are Grassmann even and ϑ_i Grassmann odd quantities, a mixed change of variables of the form

$$\begin{aligned} x_a &= x'_a + \varepsilon f_a(x', \vartheta') \\ \vartheta_i &= \vartheta'_i + \varepsilon \phi_i(x', \vartheta') \end{aligned} \quad (\text{A.24})$$

with ε being a Grassmann odd parameter leads to a Jacobian

$$J = 1 + \varepsilon \text{str}(M) \quad (\text{A.25})$$

In this expression, the matrix M under the super trace is given by

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \frac{\partial f_a}{\partial x'_b} & -\frac{\partial f_a}{\partial \vartheta'_i} \\ \frac{\partial \phi_i}{\partial x'_a} & -\frac{\partial \phi_i}{\partial \vartheta'_j} \end{pmatrix} \quad (\text{A.26})$$

and hence

$$\text{str}(M) = \text{tr}(A) - \text{tr}(D) = \frac{\partial f_a}{\partial x'_a} + \frac{\partial \phi_i}{\partial \vartheta'_i} \quad (\text{A.27})$$

(See e.g. [15], Section 1.8.2. Note, however, that in our case ε is Grassmann odd which leads to the additional minus signs in the matrix M when ε is commuted with the derivative $\partial/\partial \vartheta$).

In our case, we have two sets of commuting variables, $\mathbf{A}^{ai}(x)$ and $\mathbf{A}'^{ai}(x)$, and two sets of anti-commuting ones, $\omega^a(x)$ and $\bar{\omega}^a(x)$. Therefore, the Jacobian is given by

$$J = 1 + \varepsilon \int dx \left[\frac{\delta s A'^{ai}(x)}{\delta A'^{ai}(x)} + \frac{\delta s \lambda'^{ai}(x)}{\delta \lambda'^{ai}(x)} + \frac{\delta s \omega'^a(x)}{\delta \omega'^a(x)} + \frac{\delta s \bar{\omega}'^a(x)}{\delta \bar{\omega}'^a(x)} \right] \quad (\text{A.28})$$

However, any of these functional derivatives vanishes as a short glance at the BRST transformed fields in Eq. (53) makes obvious: The derivative always produces a Kronecker delta that is to be contracted with the structure constants. Consequently, the Jacobian of the change of variables (49) is unity.

APPENDIX B: FEYNMAN RULES

The action, as given by Eq. (46), is

$$S[\mathbf{A}, \boldsymbol{\lambda}, \omega, \bar{\omega}] = S^{(D)}[\mathbf{A}, \boldsymbol{\lambda}] + S^{(GG)}[\mathbf{A}, \omega, \bar{\omega}], \quad (B.1)$$

with

$$S^{(D)}[\mathbf{A}, \boldsymbol{\lambda}] = \int dx \left[\sigma T \boldsymbol{\lambda} \cdot \boldsymbol{\lambda}^a - i \boldsymbol{\lambda}^a \cdot \left(\mathbf{D}^{ab} \times \mathbf{B}^b + \sigma (\dot{\mathbf{A}}^a - \frac{1}{\kappa} \mathbf{D}^{ab} \nabla \cdot \mathbf{A}^b) \right) \right] \quad (B.2)$$

$$S^{(GG)}[\mathbf{A}, \omega, \bar{\omega}] = \int dx \left[-\bar{\omega}^a \dot{\omega}^a + \frac{1}{\kappa} \bar{\omega}^a \mathbf{D}^{ab} \cdot \nabla \omega^b \right] \quad (B.3)$$

1. The Propagators

The free, quadratic part of the dynamical action $S^{(D)}[\mathbf{A}, \boldsymbol{\lambda}]$ can be cast into the following symmetric form reflecting the mixing that occur between the gauge field \mathbf{A} and the auxiliary field $\boldsymbol{\lambda}$

$$S_0^{(D)}[\mathbf{A}, \boldsymbol{\lambda}] = \int dx dy \frac{1}{2} (\lambda^{ai}(x), A^{ai}(x)) (\hat{\Delta}^{-1})_{ij}^{ab}(x, y) \begin{pmatrix} \lambda^{bj}(y) \\ A^{bj}(y) \end{pmatrix} \quad (B.4)$$

with the matrix

$$(\hat{\Delta}^{-1})_{ij}^{ab}(x, y) = \begin{pmatrix} (\Delta^{-1})^{(\lambda\lambda)}_{ij}^{ab}(x, y) & (\Delta^{-1})^{(\lambda A)}_{ij}^{ab}(x, y) \\ (\Delta^{-1})^{(A\lambda)}_{ij}^{ab}(x, y) & (\Delta^{-1})^{(AA)}_{ij}^{ab}(x, y) \end{pmatrix} \quad (B.5)$$

and

$$(\Delta^{-1})^{(\lambda\lambda)}_{ij}^{ab}(x, y) = 2\sigma T \delta^{ab} \delta_{ij} \delta(x - y) \quad (B.6)$$

$$(\Delta^{-1})^{(\lambda A)}_{ij}^{ab}(x, y) = -i \delta^{ab} [(+\sigma \partial_t - \Delta) \delta_{ij} + (1 - \frac{\sigma}{\kappa}) \partial_i \partial_j] \delta(x - y) \quad (B.7)$$

$$(\Delta^{-1})^{(A\lambda)}_{ij}^{ab}(x, y) = -i \delta^{ab} [(-\sigma \partial_t - \Delta) \delta_{ij} + (1 - \frac{\sigma}{\kappa}) \partial_i \partial_j] \delta(x - y) \quad (B.8)$$

$$(\Delta^{-1})^{(AA)}_{ij}^{ab}(x, y) = 0 \quad (B.9)$$

We denote by non-bold symbols combinations of time and space variables, e.g. $\delta(x - y) = \delta(t_x - t_y) \delta^{D-1}(\mathbf{x} - \mathbf{y})$. The matrix $\hat{\Delta}^{-1}$ is symmetric in the following sense

$$(\Delta^{-1})^{(FG)}_{ij}^{ab}(x, y) = (\Delta^{-1})^{(GF)}_{ji}^{ba}(y, x) \quad (B.10)$$

Hence, the matrix propagator

$$\hat{\Delta}_{ij}^{ab}(x, y) = \begin{pmatrix} \Delta^{(\lambda\lambda)ab}_{ij}(x, y) & \Delta^{(\lambda A)ab}_{ij}(x, y) \\ \Delta^{(A\lambda)ab}_{ij}(x, y) & \Delta^{(AA)abij}(x, y) \end{pmatrix} = \begin{pmatrix} \langle \lambda^{ai}(x) \lambda^{bj}(y) \rangle_0 & \langle \lambda^{ai}(x) A^{bj}(y) \rangle_0 \\ \langle A^{ai}(x) \lambda^{bj}(y) \rangle_0 & \langle A^{ai}(x) A^{bj}(y) \rangle_0 \end{pmatrix}$$

is given by its inverse (B.11)

$$\int d^D y \Delta^{(FG)ab}_{ij}(x, y) (\Delta^{-1})^{(GH)bc}_{jk}(y, z) = \delta^{ac} \delta_{ik} \delta^{FH} \delta^D(x - z) \quad (\text{B.12})$$

or equivalently

$$\Delta^{(FG)ab}_{ij}(k) (\Delta^{-1})^{(GH)bc}_{jk}(k) = \delta^{ac} \delta_{ik} \delta^{FH} \quad (\text{B.13})$$

for the momentum space functions

$$\Delta^{(FG)ab}_{ij}(x, y) = \int \frac{d^D k}{(2\pi)^D} e^{-ik(x-y)} \Delta^{(FG)ab}_{ij}(k) \quad (\text{B.14})$$

$$(\Delta^{-1})^{(FG)ab}_{ij}(x, y) = \int \frac{d^D k}{(2\pi)^D} e^{-ik(x-y)} (\Delta^{-1})^{(FG)ab}_{ij}(k) \quad (\text{B.15})$$

Note again that though we are most of the time dealing with three-vectors, in the FOURIER transform we use four-vector notation, i.e. $e^{-ik(x-y)} = e^{-ik_0(x_0-y_0)+i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}$ leading to

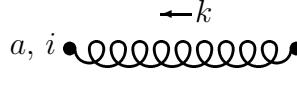
$$(\hat{\Delta}^{-1})^{ab}_{ij}(k) = \begin{pmatrix} 2\sigma T \delta^{ab} \delta_{ij} & -i \delta^{ab} [(-i\sigma k_0 + \mathbf{k}^2) \delta_{ij} - (1 - \frac{\sigma}{\kappa}) k_i k_j] \\ -i \delta^{ab} [(+i\sigma k_0 + \mathbf{k}^2) \delta_{ij} - (1 - \frac{\sigma}{\kappa}) k_i k_j] & 0 \end{pmatrix}. \quad (\text{B.16})$$

In momentum space, the gauge/auxiliary field propagators are given by:

$$a, i \text{---} \overset{\leftarrow}{k} \text{---} b, j \quad \Delta^{(\lambda\lambda)ab}_{ij}(k) = 0$$

$$a, i \text{---} \overset{\leftarrow}{k} \text{---} b, j \quad \Delta^{(\lambda A)ab}_{ij}(k) = \frac{i\delta^{ab}}{+i\sigma k_0 + |\mathbf{k}|^2} \left[\delta_{ij} + \left(1 - \frac{\sigma}{\kappa}\right) \frac{k_i k_j}{+i\sigma k_0 + \frac{\sigma}{\kappa} |\mathbf{k}|^2} \right]$$

$$a, i \text{---} \overset{\leftarrow}{k} \text{---} b, j \quad \Delta^{(A\lambda)ab}_{ij}(k) = \frac{i\delta^{ab}}{-i\sigma k_0 + |\mathbf{k}|^2} \left[\delta_{ij} + \left(1 - \frac{\sigma}{\kappa}\right) \frac{k_i k_j}{-i\sigma k_0 + \frac{\sigma}{\kappa} |\mathbf{k}|^2} \right]$$



$$\Delta^{(AA)ab}_{ij}(k) = \frac{2\sigma T \delta^{ab}}{\sigma^2 k_0^2 + |\mathbf{k}|^4} \left[\delta_{ij} + \left(1 - \frac{\sigma^2}{\kappa^2}\right) \frac{k_i k_j |\mathbf{k}|^2}{\sigma^2 k_0^2 + \frac{\sigma^2}{\kappa^2} |\mathbf{k}|^4} \right]$$

For the gauge ghosts, we have the corresponding contribution to the action, Eq. (B.3), comprises the free part

$$S_0^{(\text{GG})}[\omega, \bar{\omega}] = \int dx \bar{\omega}^a \left(-\partial_t + \frac{1}{\kappa} \Delta \right) \omega^a \quad (\text{B.17})$$

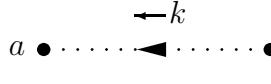
and therefore

$$(\Delta^{-1})^{(\omega)ab}(x, y) = \delta^{ab} \left(-\partial_t + \frac{1}{\kappa} \Delta \right) \delta(x - y) \quad (\text{B.18})$$

or in momentum space

$$(\Delta^{-1})^{(\omega)ab}(k) = \delta^{ab} \left(ik_0 - \frac{1}{\kappa} |\mathbf{k}|^2 \right) \quad (\text{B.19})$$

Hence the gauge ghost propagator is given by



$$\Delta^{(\omega)ab}(k) = \frac{\kappa \delta^{ab}}{i\kappa k_0 - |\mathbf{k}|^2}$$

2. The Vertices

For the interacting part of the dynamical action (B.2) we have

$$S_{\text{int}}^{(\text{D})}[\mathbf{A}, \boldsymbol{\lambda}] = \int dx \left\{ -ig f^{abc} \lambda^{ai} \left[\left(1 - \frac{\sigma}{\kappa} \right) A^{bi} \partial_j A^{cj} + 2A^{cj} \partial_j A^{bi} + A^{bj} \partial_i A^{cj} \right] - ig^2 f^{abc} f^{bde} \lambda^{ai} A^{cj} A^{dj} A^{ei} \right\} \quad (\text{B.20})$$

Thus, the theory provides a 3-point vertex containing one auxiliary and two gauge fields and a 4-point vertex of three gauge fields and one auxiliary field. To simplify explicit calculations, it is useful to symmetrise the vertices with respect to the two and three gauge fields in either case. Splitting $S_{\text{int}}^{(\text{D})}[\mathbf{A}, \boldsymbol{\lambda}]$ into the contributions corresponding to the 3- and 4-point vertex

$$S_{\text{int}}^{(\text{D})}[\mathbf{A}, \boldsymbol{\lambda}] = S_{\text{int},3}^{(\text{D})}[\mathbf{A}, \boldsymbol{\lambda}] + S_{\text{int},4}^{(\text{D})}[\mathbf{A}, \boldsymbol{\lambda}] \quad (\text{B.21})$$

one obtains

$$\begin{aligned}
 S_{\text{int},3}^{(\text{D})}[\mathbf{A}, \boldsymbol{\lambda}] = & \int dx \frac{1}{2!} (-ig) f^{abc} \lambda^{ai} \left\{ \left(1 - \frac{\sigma}{\kappa}\right) [\delta^{ij} A^{bj} \partial_k A^{ck} - \delta^{ik} A^{cj} \partial_j A^{bj}] \right. \\
 & + 2 [\delta^{ij} A^{ck} \partial_k A^{bj} - \delta^{ik} A^{bj} \partial_j A^{ck}] \\
 & \left. + [\delta^{jk} A^{bj} \partial_i A^{ck} - \delta^{kj} A^{ck} \partial_i A^{bj}] \right\} \quad (\text{B.22})
 \end{aligned}$$

$$S_{\text{int},4}^{(\text{D})}[\mathbf{A}, \boldsymbol{\lambda}] = \int dx \frac{1}{3!} (-ig^2) V_{ijkl}^{abcd} \lambda^{ai} A^{bj} A^{ck} A^{dl} \quad (\text{B.23})$$

where

$$\begin{aligned}
 V_{ijkl}^{abcd} = & f^{ace} f^{bde} (\delta^{ij} \delta^{kl} - \delta^{il} \delta^{kj}) \\
 & + f^{abe} f^{cde} (\delta^{ik} \delta^{jl} - \delta^{il} \delta^{jk}) \\
 & + f^{ade} f^{bce} (\delta^{ij} \delta^{kl} - \delta^{ik} \delta^{jl}) \quad (\text{B.24})
 \end{aligned}$$

Observing that there is an additional minus sign because we have $-S^{(\text{D})}[\mathbf{A}, \boldsymbol{\lambda}]$ in the exponent of the generating functional and noting our conventions of the FOURIER transform (B.14) of the propagators, we find for the 3-point vertex in momentum space that is symmetrised with respect to the two \mathbf{A} fields

$\lambda \mathbf{A}^2$ vertex

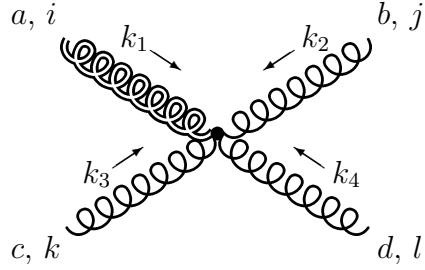
$$\begin{aligned}
 -g V_{ijk}^{abc}(\mathbf{k}_2, \mathbf{k}_3) = & -g f^{abc} \left\{ \left(1 - \frac{\sigma}{\kappa}\right) (\delta^{ij} k_3^k - \delta^{ik} k_2^j) \right. \\
 & + 2(\delta^{ij} k_2^k - \delta^{ik} k_3^j) \\
 & \left. + \delta^{jk} (k_3^i - k_2^i) \right\}
 \end{aligned}$$

Momentum conservation is thereby to be understood. By construction, the object $V_{ijk}^{abc}(\mathbf{k}_2, \mathbf{k}_3)$ is symmetric in the last two pairs of indices (and corresponding momenta), i.e.

$$V_{ijk}^{abc}(\mathbf{k}_2, \mathbf{k}_3) = V_{ikj}^{acb}(\mathbf{k}_3, \mathbf{k}_2) \quad (\text{B.25})$$

Analogously, the symmetrised 4-point vertex is found to be

$\lambda \mathbf{A}^3$ vertex



$$ig^2 V_{ijkl}^{abcd} = ig^2 \left\{ \begin{aligned} & f^{ace} f^{bde} (\delta^{ij} \delta^{kl} - \delta^{il} \delta^{kj}) \\ & + f^{abe} f^{cde} (\delta^{ik} \delta^{jl} - \delta^{il} \delta^{jk}) \\ & + f^{ade} f^{bce} (\delta^{ij} \delta^{kl} - \delta^{ik} \delta^{jl}) \end{aligned} \right\}$$

where V_{ijkl}^{abcd} was already introduced in Eq. (B.24) and is symmetric in the last three pairs of indices

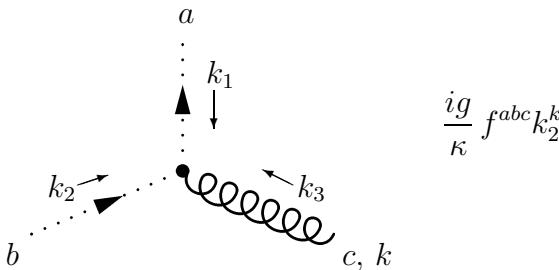
$$V_{ijkl}^{abcd} = V_{ijlk}^{abdc} = V_{ikjl}^{acbd} = V_{iklj}^{acdb} = V_{iljk}^{adbc} = V_{ilkj}^{adcb} \quad (\text{B.26})$$

For the ghost sector, the corresponding interaction term extracted from Eq. (B.3) is given by

$$S_{\text{int}}^{(\text{GG})}[\mathbf{A}, \omega, \bar{\omega}] = \int dx \frac{(-g)}{\kappa} f^{abc} \bar{\omega}^a (\mathbf{A}^c \cdot \nabla) \omega^b = \int dx \frac{(-g)}{\kappa} f^{abc} \bar{\omega}^a A^{ck} \partial_k \omega^b \quad (\text{B.27})$$

and leads to the momentum space vertex

$\omega \bar{\omega} \mathbf{A}$ vertex



APPENDIX C: EXPLICIT CONSEQUENCES OF IDENTITIES TO LOWER N-POINT FUNCTIONS

We will now find explicit identities for the lower n-point function from the identities obtained in Section III.

1. 1-point Functions

Let us start by explicitly writing down the consequences of Ghost number conservation, Eqs. (85) – (87), to the one-point functions of the theory. Taking the functional derivative of Eq. (86) with respect to one of the sources J_ω , $J_{\bar{\omega}}$, I_{sA} , $I_{s\lambda}$ or $I_{s\omega}$ and evaluating for $J = I = 0$ yields

$$\begin{aligned} \frac{\delta W[J, I]}{\delta J_\omega^a(x)} \bigg|_{J=I=0} &= 0 & \frac{\delta W[J, I]}{\delta J_{\bar{\omega}}^a(x)} \bigg|_{J=I=0} &= 0 & \frac{\delta W[J, I]}{\delta I_{sA}^{ai}(x)} \bigg|_{J=I=0} &= 0 \\ \frac{\delta W[J, I]}{\delta I_{s\lambda}^{ai}(x)} \bigg|_{J=I=0} &= 0 & \frac{\delta W[J, I]}{\delta I_{s\omega}^a(x)} \bigg|_{J=I=0} &= 0 \end{aligned} \quad (\text{C.1})$$

The same relations follow for the derivatives of $Z[J, I]$ from Eq. (85). On the other hand, Eq. (64) implies

$$\frac{\delta \Gamma}{\delta A^{ai}(x)} \bigg|_{J=I=0} = 0 \quad \frac{\delta \Gamma}{\delta \lambda^{ai}(x)} \bigg|_{J=I=0} = 0 \quad \frac{\delta \Gamma}{\delta \omega^a(x)} \bigg|_{J=I=0} = 0 \quad \frac{\delta \Gamma}{\delta \bar{\omega}^a(x)} \bigg|_{J=I=0} = 0 \quad (\text{C.2})$$

and the combination of Eq. (65) and (C.1) gives

$$\frac{\delta \Gamma}{\delta I_{sA}^{ai}(x)} \bigg|_{J=I=0} = 0 \quad \frac{\delta \Gamma}{\delta I_{s\lambda}^{ai}(x)} \bigg|_{J=I=0} = 0 \quad \frac{\delta \Gamma}{\delta I_{s\omega}^a(x)} \bigg|_{J=I=0} = 0 \quad (\text{C.3})$$

The last first derivative of Γ can be computed from the stochastic Ward identities, Eq. (78)

$$\frac{\delta \Gamma}{\delta I_{s\bar{\omega}}^a(x)} \bigg|_{J=I=0} = 0 \quad (\text{C.4})$$

Thus, all first derivatives of Γ have to vanish.

2. 2-point Functions

The consequences of ghost number conservation to the second derivatives of $Z[J, I]$ and $W[J, I]$ are summarised in the following table, indicating for any pair of sources whether the corresponding second derivative (evaluated for $J = I = 0$) is restricted to vanish or not by

ghost number conservation

	J_A	J_λ	J_ω	$J_{\bar{\omega}}$	I_{sA}	$I_{s\lambda}$	$I_{s\omega}$	$I_{s\bar{\omega}}$
J_A			0	0	0	0	0	
J_λ			0	0	0	0	0	
J_ω	0	0	0		0	0	0	0
$J_{\bar{\omega}}$	0	0		0			0	0
I_{sA}	0	0	0		0	0	0	0
$I_{s\lambda}$	0	0	0		0	0	0	0
$I_{s\omega}$	0	0	0	0	0	0	0	0
$I_{s\bar{\omega}}$			0	0	0	0	0	

(C.5)

The analogous result for the second derivatives of the 1PI generating functional $\Gamma[\mathbf{A}, \boldsymbol{\lambda}, \omega, \bar{\omega}; I]$ (as well evaluated for vanishing sources $J = I = 0$) is

	A	λ	ω	$\bar{\omega}$	I_{sA}	$I_{s\lambda}$	$I_{s\omega}$	$I_{s\bar{\omega}}$
A			0	0	0	0	0	0
λ			0	0	0	0	0	
ω	0	0	0				0	0
$\bar{\omega}$	0	0		0	0	0	0	0
I_{sA}	0	0		0	0	0	0	0
$I_{s\lambda}$	0	0		0	0	0	0	0
$I_{s\omega}$	0	0	0	0	0	0	0	0
$I_{s\bar{\omega}}$			0	0	0	0	0	

(C.6)

In the following, we will often rely on the information summarised in these tables dropping certain terms that are bound to zero by ghost number conservation from our calculations without further notice. To start with, let us recall the gauge Ward identity in terms of the generating functional of connected correlation functions $W[J, I]$. It was found in Eq. (61) to read

$$\int dx \left[J_A^{ai}(x) \frac{\delta W[J, I]}{\delta I_{sA}^{ai}(x)} + J_\lambda^{ai}(x) \frac{\delta W[J, I]}{\delta I_{s\lambda}^{ai}(x)} + J_\omega^{ai}(x) \frac{\delta W[J, I]}{\delta I_{s\omega}^{ai}(x)} + J_{\bar{\omega}}^{ai}(x) \frac{\delta W[J, I]}{\delta I_{s\bar{\omega}}^{ai}(x)} \right] = 0 \quad (\text{C.7})$$

Taking second derivatives, a variety of possibilities arise. For instance, choosing $\delta/\delta J_A^{ai}(x)$ and $\delta/\delta J_A^{bj}(y)$ yields after setting sources to zero

$$\frac{\delta^2 W[J, I]}{\delta J_A^{bj}(y) \delta I_{sA}^{ai}(x)} \bigg|_{J=I=0} + \frac{\delta^2 W[J, I]}{\delta J_A^{ai}(x) \delta I_{sA}^{bj}(y)} \bigg|_{J=I=0} = 0 \quad (\text{C.8})$$

However, due to ghost number conservation both of these terms are zero by themselves. Likewise, the combination of $\delta/\delta J_A^{ai}(x)$ with $\delta/\delta J_\lambda^{bj}(y)$ or $\delta/\delta J_\omega^b(y)$ does not lead to any new relation when ghost number conservation is taken into account. The fourth possibility however, combining $\delta/\delta J_A^{ai}(x)$ and a derivative with respect to $J_{\bar{\omega}}^b(y)$, results in the identity

$$\frac{\delta^2 W[J, I]}{\delta J_{\bar{\omega}}^b(y) \delta I_{sA}^{ai}(x)} \Big|_{J=I=0} + \frac{\delta^2 W[J, I]}{\delta J_A^{ai}(x) \delta I_{s\bar{\omega}}^b(y)} \Big|_{J=I=0} = 0 \quad (\text{C.9})$$

that will be further exploited in a moment. Considering the combinations of $\delta/\delta J_\lambda^{ai}(x)$ with one of the derivatives $\delta/\delta J_\lambda^{bj}(y)$ or $\delta/\delta J_\omega^b(y)$ again only leads to trivial relations in view of ghost number conservation. The pairing of $\delta/\delta J_\lambda^{ai}(x)$ with $\delta/\delta J_{\bar{\omega}}^b(y)$ yields

$$\frac{\delta^2 W[J, I]}{\delta J_{\bar{\omega}}^b(y) \delta I_{s\lambda}^{ai}(x)} \Big|_{J=I=0} + \frac{\delta^2 W[J, I]}{\delta J_\lambda^{ai}(x) \delta I_{s\bar{\omega}}^b(y)} \Big|_{J=I=0} = 0 \quad (\text{C.10})$$

However, this relation is a consequence of the two simpler identities

$$\frac{\delta^2 W[J, I]}{\delta J_{\bar{\omega}}^b(y) \delta I_{s\lambda}^{ai}(x)} \Big|_{J=I=0} = 0 \quad \frac{\delta^2 W[J, I]}{\delta J_\lambda^{ai}(x) \delta I_{s\bar{\omega}}^b(y)} \Big|_{J=I=0} = 0 \quad (\text{C.11})$$

induced by the stochastic Ward identity (75). The remaining possibilities finally, choosing two derivatives with respect to ω , two derivatives with respect to $\bar{\omega}$, or one with respect to ω , one to $\bar{\omega}$ again express ghost number conservation only. Hence, up to the level of second derivatives Eq. (C.9) is the only restriction imposed by the gauge BRST symmetry beyond relations that already follow from the stochastic Ward identity or simply are a consequence of ghost number conservation. Implications of the stochastic Ward identities (75) and (78) are most importantly the vanishing of the auxiliary field propagator to all orders

$$G^{(\lambda\lambda)ab}_{ij}(x, y) = \frac{\delta^2 W[J, I]}{\delta J_\lambda^{ai}(x) \delta J_\lambda^{bj}(y)} \Big|_{J=I=0} = 0 \quad (\text{C.12})$$

or, equivalently, of the (AA) self-energy component

$$\Pi^{(AA)ab}_{ij}(x, y) = \frac{\delta^2 \Gamma}{\delta A^{ai}(x) \delta A^{bj}(y)} \Big|_{J=I=0} - (\Delta^{-1})^{(AA)ab}_{ij}(x, y) = 0 \quad (\text{C.13})$$

where in addition to Eq. (78) it was used that the (AA) component of the inverse free propagator is zero too (cf. Eq. (B.9)). Note that Eq. (C.13) is a special case of the general statement that there are no pure gauge field vertices in the theory: All proper vertex functions of the form

$$\Gamma^{(AA\dots A)ab\dots c}_{ij\dots k}(x, y, \dots, z) = \frac{\delta^n \Gamma}{\delta A^{ai}(x) \delta A^{bj}(y) \dots \delta A^{ck}(z)} \Big|_{J=I=0} \quad (\text{C.14})$$

vanish as an immediate consequence of the stochastic Ward identity (78). Further implications up to second derivatives (neglecting those only expressing ghost number conservation) are

$$\frac{\delta^2 \Gamma}{\delta \omega^b(y) \delta I_{s\lambda}^{ai}(x)} \Big|_{J=I=0} = 0 \quad \frac{\delta^2 \Gamma}{\delta A^{ai}(x) \delta I_{s\bar{\omega}}^b(y)} \Big|_{J=I=0} = 0 \quad (\text{C.15})$$

together with the equivalent identities (C.11),

$$\frac{\delta \Gamma}{\delta I_{s\omega}^a(x)} \Big|_{J=I=0} = - \frac{\delta W}{\delta I_{s\omega}^a(x)} \Big|_{J=I=0} = 0 \quad (\text{C.16})$$

and for completeness finally

$$\frac{\delta^2 \Gamma}{\delta I_{s\bar{\omega}}^a(x) \delta I_{s\bar{\omega}}^b(y)} \Big|_{J=I=0} = - \frac{\delta^2 W}{\delta I_{s\bar{\omega}}^a(x) \delta I_{s\bar{\omega}}^b(y)} \Big|_{J=I=0} = 0 \quad (\text{C.17})$$

This last identity, however, does not lead to a simple relation among the lower n-point functions because both of the derivatives act on sources of the BRST transformed fields. In general, to make sense of the above identities we will have to translate the derivatives of the I -type to such with respect to sources of the fundamental fields. For instance, one has

$$\begin{aligned} \frac{\delta Z}{\delta I_{sA}^{ai}(x)} &= \int \mathcal{D}\mathbf{A} \mathcal{D}\boldsymbol{\lambda} \mathcal{D}\omega \mathcal{D}\bar{\omega} (D_i^{ab}(x) \omega^b(x)) \exp\{(\dots)\} \\ &= \int \mathcal{D}\mathbf{A} \mathcal{D}\boldsymbol{\lambda} \mathcal{D}\omega \mathcal{D}\bar{\omega} \left(\partial_i \left(-\frac{\delta}{\delta J_\omega^a(x)} \right) - g f^{abc} \frac{\delta}{\delta J_A^{ci}(x)} \left(-\frac{\delta}{\delta J_\omega^b(x)} \right) \right) \exp\{(\dots)\} \\ &= -\partial_i \frac{\delta Z}{\delta J_\omega^a(x)} + g f^{abc} \frac{\delta^2 Z}{\delta J_A^{ci}(x) \delta J_\omega^b(x)} \end{aligned} \quad (\text{C.18})$$

where the dots abbreviate the usual exponent of the generating functional as given in Eq. (56). Expressing this identity in terms of $W = \ln Z$ yields

$$\frac{\delta W}{\delta I_{sA}^{ai}(x)} = -\partial_i \frac{\delta W}{\delta J_\omega^a(x)} + g f^{abc} \left(\frac{\delta^2 W}{\delta J_A^{ci}(x) \delta J_\omega^b(x)} + \frac{\delta W}{\delta J_A^{ci}(x)} \frac{\delta W}{\delta J_\omega^b(x)} \right) \quad (\text{C.19})$$

Analogously, one obtains after some algebra

$$\frac{\delta W}{\delta I_{s\lambda}^{ai}(x)} = g f^{abc} \left(\frac{\delta^2 W}{\delta J_\lambda^{ci}(x) \delta J_\omega^b(x)} + \frac{\delta W}{\delta J_\lambda^{ci}(x)} \frac{\delta W}{\delta J_\omega^b(x)} \right) \quad (\text{C.20})$$

$$\frac{\delta W}{\delta I_{s\omega}^a(x)} = \frac{1}{2} g f^{abc} \left(\frac{\delta^2 W}{\delta J_\omega^c(x) \delta J_\omega^b(x)} + \frac{\delta W}{\delta J_\omega^c(x)} \frac{\delta W}{\delta J_\omega^b(x)} \right) \quad (\text{C.21})$$

$$\begin{aligned} \frac{\delta W}{\delta I_{s\bar{\omega}}^a(x)} &= i\sigma \partial_i \frac{\delta W}{\delta J_\lambda^{ai}(x)} - i\sigma g f^{abc} \left(\frac{\delta^2 W}{\delta J_A^{ci}(x) \delta J_\lambda^{bi}(x)} + \frac{\delta W}{\delta J_A^{ci}(x)} \frac{\delta W}{\delta J_\lambda^{bi}(x)} \right) \\ &\quad + g f^{abc} \left(\frac{\delta^2 W}{\delta J_\omega^c(x) \delta J_\omega^b(x)} + \frac{\delta W}{\delta J_\omega^c(x)} \frac{\delta W}{\delta J_\omega^b(x)} \right) \end{aligned} \quad (\text{C.22})$$

With these substitutions Eq. (C.9) translates to

$$\begin{aligned} \partial_i G^{(\omega)ab}(x, y) - i\sigma \partial_j G^{(A\lambda)ab}_{ij}(x, y) = \\ -gf^{acd}W^{(\bar{\omega}\omega A)bcd}_i(y, x, x) - i\sigma gf^{bcd}W^{(AA\lambda)acd}_{ijj}(x, y, y) + gf^{bcd}W^{(\bar{\omega}\omega A)cda}_i(y, y, x) \end{aligned} \quad (\text{C.23})$$

To further proceed, we express the connected three-point functions by their 1PI counterparts and transform into momentum space. Especially note that we pull out the momentum conserving delta function from the definition of our proper vertices. Hence, only $N - 1$ momentum variables appear in the argument of a N -point vertex. For instance, we use $\Gamma^{(\bar{\omega}\omega G)abc}_{j}(k_1, k_2)$ where the superscript G is either the gauge field A or the auxiliary field λ and k_1 and k_2 refer to the (incoming) momenta along the ghost lines leaving and entering the vertex in this order. Accordingly, in $\Gamma^{(FGH)abc}_{ijk}(k_2, k_3)$ with $F, G, H \in \{A, \lambda\}$ the two arguments k_2 and k_3 refer to the incoming momenta along the G and H line respectively. With these definitions, Eq. (C.23) takes the form

$$\begin{aligned} ik^i G^{(\omega)ab}(k) + \sigma k^j G^{(A\lambda)ab}_{ij}(k) = \\ + G^{(\omega)b'b}(k) \int \frac{d^D k'}{(2\pi)^D} gf^{acd} G^{(\omega)cc'}(k') G^{(AF)dd'}_{i'i'}(k - k') \Gamma^{(\bar{\omega}\omega F)c'b'd'}_{i'}(-k', k) \\ - G^{(AF)aa'}_{i'i'}(k) \int \frac{d^D k'}{(2\pi)^D} gf^{bcd} \left[G^{(\omega)c'c}(k') G^{(\omega)dd'}(k' - k) \Gamma^{(\bar{\omega}\omega F)d'c'a'}_{i'}(k - k', k') \right. \\ \left. - i\sigma G^{(A\lambda)c'c}_{j'j}(k') G^{(AG)dd'}_{jk'}(k' - k) \Gamma^{(FGA)a'd'c'}_{i'k'j'}(k - k', k') \right] \end{aligned} \quad (\text{C.24})$$

The indices F and G in this equation are summation indices taking the two values A and λ . However, as we will show now, the stochastic Ward identity leads to a cancellation among some of the terms involved. To this end, let us express also the identities derived from the stochastic Ward identity in the language of full propagators and proper vertex functions. As mentioned above, identity (C.16) relates the normalisations of the gauge ghost and mixed auxiliary/gauge field propagator

$$gf^{abc} \int \frac{d^D k}{(2\pi)^D} [G^{(\omega)cb}(k) - i\sigma G^{(A\lambda)cb}_{ii}(k)] = 0 \quad (\text{C.25})$$

From the first of the Eqs. (C.15) one obtains after some relabelling

$$\int \frac{d^D k'}{(2\pi)^D} gf^{bcd} G^{(A\lambda)c'c}_{i'i}(k') G^{(\omega)dd'}(k' - k) \Gamma^{(\bar{\omega}\omega A)d'ac'}_{i'}(k - k', -k) = 0 \quad (\text{C.26})$$

from the second equation

$$\int \frac{d^D k'}{(2\pi)^D} g f^{bcd} G^{(\omega)c'c}(k') G^{(\omega)dd'}(k' - k) \Gamma^{(\bar{\omega}\omega A)d'c'a'}(k - k', k') - i\sigma \int \frac{d^D k'}{(2\pi)^D} g f^{bcd} G^{(A\lambda)c'c}_{j'j}(k') G^{(A\lambda)dd'}_{jk'}(k' - k) \Gamma^{(A\lambda A)a'd'c'}_{ik'j'}(k - k', k') = 0 \quad (\text{C.27})$$

Here we have used $\Gamma^{(FGH)abc}_{ijk}(k_2, k_3) = \Gamma^{(GHF)bca}_{jki}(k_3, -k_2 - k_3)$ in accordance with our definition of the vertex functions. Let us now come back to Eq. (C.24), that was found to be the expression of the gauge Ward identity on the level of second derivatives. With the summation index F taking the value A , the second integral in Eq. (C.24) consists of three terms: the one with the two ghost propagators and two copies of the second term corresponding to the two possible values $G = \lambda$ and $G = A$. The last of these terms is zero because it contains $\Gamma^{(AAA)}$. Moreover, the remaining two terms cancel each other due to Eq. (C.27) as a consequence of the stochastic Ward identity. Hence, there is only a contribution of the second integral in Eq. (C.24) for $F = \lambda$. The first integral, however, contributes for both choices $F = \lambda$ and $F = A$ (and likewise if F is set to λ in the second integral, G can still take both values $G = \lambda, A$).

The gauge BRST symmetry therefore leads to the following identity to be obeyed by the full propagators and proper vertex functions of the theory

$$\begin{aligned} ik^i G^{(\omega)ab}(k) + \sigma k^j G^{(A\lambda)ab}_{ij}(k) = & \\ & + G^{(\omega)b'b}(k) \int \frac{d^D k'}{(2\pi)^D} g f^{acd} G^{(\omega)cc'}(k') G^{(AA)dd'}_{i'j'}(k - k') \Gamma^{(\bar{\omega}\omega A)c'b'd'}_{i'j'}(-k', k) \\ & + G^{(\omega)b'b}(k) \int \frac{d^D k'}{(2\pi)^D} g f^{acd} G^{(\omega)cc'}(k') G^{(A\lambda)dd'}_{i'j'}(k - k') \Gamma^{(\bar{\omega}\omega\lambda)c'b'd'}_{i'j'}(-k', k) \\ & - G^{(A\lambda)aa'}_{i'j'}(k) \int \frac{d^D k'}{(2\pi)^D} g f^{bcd} \left[G^{(\omega)c'c}(k') G^{(\omega)dd'}(k' - k) \Gamma^{(\bar{\omega}\omega\lambda)d'c'a'}_{i'j'}(k - k', k') \right. \\ & \quad - i\sigma G^{(A\lambda)c'c}_{j'j}(k') G^{(AA)dd'}_{jk'}(k' - k) \Gamma^{(\lambda\lambda A)a'd'c'}_{i'k'j'}(k - k', k') \\ & \quad \left. - i\sigma G^{(A\lambda)c'c}_{j'j}(k') G^{(A\lambda)dd'}_{jk'}(k' - k) \Gamma^{(\lambda\lambda A)a'd'c'}_{i'k'j'}(k - k', k') \right] \quad (\text{C.28}) \end{aligned}$$

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